OPTIMAL AND SUBOPTIMAL FILTERING FOR TIME-INVARIANT SYSTEMS EXCITED BY COMPOUND POISSON PROCESSES

By

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The problem addressed in this research is the estimation of a linear time-invariant scalar system excited by a compound Poisson process and observed imperfectly through measurements with additive continuous Gaussian errors. Previous results of Eckberg, Kwakernaak, and Au for this problem have been inconclusive or inappropriately concluded in determining that a linear filter, Kalman or Wiener, performs as adequately as a truncated optimal infinite-dimensional non-linear filter. The contribution of this research is the determination conditions, as specified by the system parameters, under which the linear filter performs as adequately as the truncated non-linear filter and conditions under which this non-linear filter has better performance than the linear filter.

Specifically, for systems characterized by high signal-to-noise ratios, improved performance, as determined by the mean squared error, can be

obtained by using non-linear filtering equations. This improved performance can be achieved by implementing a constraint of the positive semi-definiteness of the second moment for stable systems and positive definiteness for unstable systems. The filtering equations based on a fourth quasimoment truncation with the second moment constraint exhibit the best performance.

CHAPTER ONE
INTRODUCTION

Background

The research presented in this dissertation is the optimal and suboptimal filtering of continuous linear systems driven by compound Poisson white noise with measurements corrupted by additive Gaussian white noise. Applications using stochastic models such as these are found in many fields. The specific applications of interest and which motivated this research are those found in the aerospace applications. These will be discussed following the review of other applications.

In the communications field, this model has been used for low-frequency atmospheric noise (e.g., Eckberg, 1970, and Kwakernaak, 1980). Environmental problems such as the occasional polluting of a river have also been modeled by this type stochastic process (e.g., Kwakernaak, 1975). Random system failures in the field of system reliability also adapt to this model (e.g., Willsky and Jones, 1976, and Fiorina and Maffezzoni, 1979). Finally by analyzing seismic signals oil exploration is conducted. The seismic signals are represented as the output of such a model (e.g., Mendel, 1977). In certain of these applications, post data collection processing is acceptable. In these instances optimal smoothing rather than filtering has been used (e.g., Au, 1979, Au and Haddad, 1978, and Au, Haddad and Poor, 1982).

Optimal estimation via filtering is the desired approach since the applications of interest for this research require real time application

of control for corrective action. An application for this filtering approach is the state estimation of tracked randomly maneuvering targets. Randomly maneuvering target acceleration has been modeled as a random telegraph wave process (e.g., Price and Warren, 1973, Zarchan, 1979, and Rogers, 1982). Modeled as a random telegraph wave, the target acceleration is a maximum effort fixed level which alternates in direction and the level is achieved instantaneously. In these papers, this process is realized as a first order continuous Gaussian process for implementation into a conventional Kalman filter. Other models which include varying amplitudes, response time and time duration may be more appropriate. These models would be determined with system impulse response characteristics tailored to the maneuvering target's characteristics and have the random time switching characteristics.

An alternate approach to target modeling and estimation has been used in which the target maneuvers randomly in time and at random amplitudes (a compound Poisson process) with the amplitude levels limited to discrete levels and number of levels (e.g., Moose, 1975, Gholson and Moose, 1979, and Moose, Vanlandingham, and McCabe, 1979). This approach assumes that the time spent at a maneuver amplitude level is greater than the dynamic response time of the Kalman filter. It has been shown (Gholson and Moose, 1979), that increasing the number of discrete maneuver levels so as to approaching a continuous function improves this estimator's performance.

The various approaches summarized here use assumptions which permit the use of the theoretical framework of Kalman filtering rather than formulating the problem as the conditional mean estimate of a continuous system driven impulsively by a compound Poisson process with continuous

measurements. Previous attempts at the use of the conditional mean estimate to be developed in this paper have not been successful. In this dissertation, the filtering equations for a system driven by a compound Poisson process and measured with additive Gaussian noise will be presented, and, where previous attempts have failed, conditions in which improvements are achievable relative to the linear filtering approach will be defined.

Statement of the Problem

The system to be estimated is described by the following stochastic differential equation:

$$dx_t = a\ x_t\ dt + d\sigma_t. \qquad 1$$

This system is driven by the process σ_t which is a compound Poisson process with intensity λ and zero-mean jump amplitudes governed by the probability density p_α, where α symbolically represents the moment of the distribution, i.e., α_1 is mean, α_2 is variance, etc. The probability that k jumps occur up to time t is given by

$$P(n=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},\ t > 0. \qquad 2$$

The process x_t has discontinuous sample paths.

This process is observed by the measurement process, y_t, where

$$dy_t = c\ x_t\ dt + dv_t. \qquad 3$$

The Brownian motion process, v_t, has the variance $E(dvdv) = rdt$. The process σ_t and v_t are assumed independent. The problem addressed here assumes that the jumps are not detected or observed as part of the

measurement process. It is also assumed that the model (e.g., Johnson, 1977), and the associated statistical parameters have been determined.

Previous work (e.g., Fisher, 1967; Fisher and Stear, 1967; Eckberg, 1970; Kwakernaak, 1980; and Au, 1979) developed the optimal filter for this system and it is an infinite set of coupled non-linear stochastic differential equations. Simulation techniques (Eckberg, 1970; and Kwakernaak, 1975) have been used to evaluate the performance of truncated equations at various truncation points. Conclusions drawn from these simulations have been unsuccessful in determining whether improvements are obtainable with the additional equations over those provided by a linear or Kalman filter.

The problem addressed here is the construction of an approximate solution to the optimal filtering equations; the solution is obtained from an infinite set of optimal filtering equations. Additionally, conditions under which the truncated (suboptimal) equations exhibit improved performance over the linear or Kalman filter are determined. It will be shown that combinations of the system parameters a, λ, α, c and r govern the filter response and the potential for improvement over the linear filtering equations.

These results will be accomplished by establishing the optimal filtering equations using quasimoment functions. Also, limit arguments are used to establish when the infinite set of equations decouple and become the same as the linear equations. The numerical example problems examined in the previous simulation studies will be used in this research to demonstrate the new conditions established for better performance of the solution to the truncated non-linear equations over the linear filtering equations.

Review of Previous Work

Previous developments of optimal filtering equations for the system in equation (1) include Wonham (1965), Fisher (1967), Eckberg (1978), and Kwakernaak (1975). Wonham obtained an optimal estimator for the random telegraph wave. Using the conditional density function, Wonham obtained a non-linear finite dimensional filter. By examination of the stationary error variance of the non-linear filter and of the standard linear Weiner filter, Wonham concluded that the quantity composed of the product of the process switching rate and the measurement noise variance amplification can be used as a condition to judge the non-linear filter's performance. It was shown that as this quantity became large, the stationary error variance of the two filters became equivalent, and as this quantity approached zero, the stationary error variance of the non-linear filter was less than that for the linear filter. A new combination of system parameters including those obtained by Wonham is developed for applicability of the filter equations examined in this research.

Fisher (1967) developed optimal filtering equations for a general n-dimensional non-linear system and non-linear measurements of the state, and for the case where the system was driven simultaneously by compound Poisson and Weiner processes. Filtering equations for the conditional mean, covariance, and third and fourth quasimoment functions were obtained. Quasimoment functions were used because both the characteristic function and the probability density function may be easily and directly expressed in terms of the quasimoment function; see Appendix A. Additionally, assuming higher-order quasimoment functions to be zero does not imply any unusual type density function as in the case where assuming

higher-order central moments to be zero implies that the density function contains derivatives of all orders of the Dirac delta function. From this work, it was shown that the dimensionality of the optimal filter increases rapidly with the increase of state dimension necessitated because of the quasimoment terms. Fisher concluded with recommendations for examining the stability of the truncated non-linear equations and a simulation study to determine the trade-off between truncation and estimation accuracy.

Eckberg (1970) obtained optimal filtering equations for the system of equation (1) using central moments, cumulants and quasimoments. Using the propagational equation for the conditional density function developed by Snyder (1968), and the Fourier transform properties of the probability density and the characteristic functions, the optimal filtering equations were obtained. For the scalar case, Eckberg's equations differ from those obtained by Fisher. Eckberg presented results from a simulation study of an increasing number of stochastic differential equations in the truncated filtering equations. The results presented indicated that with increasing jump amplitudes, the truncated filtering equations yielded lower mean square error than the linear filter. The linear filter had better performance for input jump amplitudes below certain levels. Instabilities with the higher moment filters seemed to be the most difficult problem in the simulation. Because of the differences between the equations obtained by Fisher and Eckberg, an alternate approach seems warranted.

Kwakernaak (1975) applied martingale theory to filtering problems of linear systems excited by Poisson white noise with observations corrupted with additive Gaussian measurement noise. For an n-dimensional system,

stochastic differential equations for the conditional density function, central moments and cumulants were derived, and two approximate methods for solving the filtering problem were developed. Numerical results were presented for a scalar system comparing filter performance for a linear filter, a non-linear filter truncated at the seventh cumulant, and a non-linear filter obtained by a Ritz-Galerkin method. Kwakernaak also reported difficulties with instabilities. The simulation results presented showed no significant improvement of the truncated cumulant filter over the linear filter. In the case examined, the cumulant filter performed better when x_t was small but deteriorated when x_t was large. The system parameters for this example were significantly different than those used by Eckberg (1970). Also, the trends in the performance of the non-linear filters with input amplitude were contrary to other results. Kwakernaak expressed an interest in establishing bounds on the possible improvement of the non-linear filter over the linear filter. This led to the conditions developed and evaluated in this research.

Recently, Au (1979). motivated by the poor filter performance reported by Kwakernaak (1975), developed a suboptimal sequential smoothing scheme. As motivation for the smoothing approach, Au showed that as the Poisson rate parameter, λ, tends to zero, a linear filter performs poorly. In this proof, the rate parameter was the only system parameter used in the limiting arguments. Not proven in these works but referred to in the work of Kwakernaak, was the statement that optimum filters perform poorly. The case of low-intensity Poisson Jump amplitudes is examined herein, and a counter example presented.

In this work, the quasimoment functions will be used to develop the optimal filter equations. Filtering equations up to the sixth

quasimoment will be developed. Filtering equations up to the fourth quasimoment will be shown to be in agreement with those of Fisher (1967). Conditions under which the truncated filtering equations approach or deviate from the linear filtering equations will be developed via the conditional characteristic function. Stationary quasimoments and limit arguments using combinations of system parameters will be used to show that the truncated non-linear filtering equations decouple and become a linear filter again under certain limits. This implies that only in the counter case, strong coupling, can the non-linear filtering equations have the potential for improved performance over the linear equations. The existence of stationary values of the quasimoments requires stability of the filtering equations. Conditions for the truncated non-linear filtering equations will be established. Finally, using the previously published numerical example problems, the conditions established for improved filter performance will be examined to confirm the performance improvements.

Dissertation Outline

In Chapter Two, the optimal filtering equations are developed for the system described by equations (1) and (3). Following the martingale approach of Kwakernaak (1975), the stochastic differential equation for the conditional characteristic function is derived. Then, departing from Kwakernaak (1975), the quasimoment function is used to develop the optimal filtering equations up to the sixth quasimoment.

In Chapter Three, the conditions of performance of the non-linear filtering equations relative to the linear filtering equations are derived.

First, the truncation of the filtering equations, via the conditional characteristic function, is performed to determine in general the condition for deviation from or convergence to the linear filtering case. Then, to serve as motivation for examining non-linear filtering equations' performance, stability and performance of the linear filter equation is examined using Lyapunov techniques. Next, the stability and decoupling of the non-linear filtering equations are investigated as the system approaches the conventional Gaussian problem for low signal-to-noise conditions. A constraint to maintain stability of the non-linear filtering equations is established using Syapunov techniques.

In Chapter Four, numerical results are presented for examples previously reported in the literature. Conditions of low and high signal-to-noise are numerically investigated and the results presented. Using the example problem of Kwakernaak (1975) and a new example problem, performance improvement of the non-linear filtering equations is demonstrated. These numerical examples confirm the analytical results including the stability constraint developed in Chapter Three.

Chapter Five presents corresponding developments for an unstable system as those presented in Chapters Three and Four.

Finally, in Chapter Six, conclusions of this work and recommendations for future work are presented. Appendix A provides a review of quasi-moment functions from Fisher (1967). The optimal filtering equation using the martingale approach from Kwakernaak (1975) is given in Appendix B. A listing of the computer program used to generate the numerical simulation results is provided in Appendix C.

CHAPTER TWO
DEVELOPMENT OF OPTIMAL FILTERING EQUATIONS

Introduction

In this chapter, the first six of the infinite sets of optimal filtering equations are derived. Following the approach of Kwakernaak (1975) in the use of martingale theory, the differentiation rule of Doleans-Dade and Meyer referred to by Kwakernaak (1975) and the fundamental filtering theorem of Kwakernaak's are used to obtain the stochastic differential equation for the conditional characteristic function. Then, departing from Kwakernaak (1975), the optimal filtering equations in terms of quasimoment functions are derived up to the sixth quasimoment. It is shown that the filtering equations up to the fourth quasimoment are in agreement with the scalar version of the results of Fisher (1967). Truncated forms of these filtering equations are examined in the following chapter as an approximate solution of the filtering problem.

Conditional Characteristic Equation

Differentiation Rule

Here the Doleans-Dade and Meyer differentiation rule will be summarized from Appendix B. Let the process x_t be a discontinuous semi-martingale, and let φ be a twice differentiable function of x_t.

Then the process $Q_t \triangleq \varphi(x_t)$ is also a semi-martingale, such that

$$dQ_t = \varphi_x(x_{t-})dx_t + \tfrac{1}{2}[\varphi_{xx}(x_{t-})d < x^c, x^c >_t] \qquad 4$$

$$+ d \sum_{t_0 \le s \le t} [\varphi(x_s) - \varphi(x_{s-}) - \varphi_x(x_{s-})(x_s - x_{s-})]$$

where φ_x and φ_{xx} are the first and second partial derivatives of the function φ, and x^c is the continuous part of x and the summation is carried out over those values of s where x jumps. The last term in equation (4) distinguishes it from the Ito differentiation rule (e.g., Jazwinski, 1970) for Gaussian driven systems. Other differentiation rules, such as the Kushner-Stratonovich equation (e.g., McGarty, 1974) for propagating the conditional probability density function, can also be used to obtain the desired equations.

Propagating the Conditional Characteristic Function

For equation (4), define the function Q_t as

$$Q_t \triangleq e^{iux_t} \qquad 5$$

where $i = \sqrt{-1}$ and u is real. Then applying equation (4) to this function results in

$$d(e^{iux_t}) = iuax_t e^{iux_t} dt + \lambda_t E(e^{iu\alpha} - 1)e^{iux_t} dt \qquad 6$$

$$+ d \sum_{t_0 \le s \le t} (e^{iux_s} - e^{iux_{s-}}) - \lambda_t E(e^{iu\alpha} - 1)e^{iux_t} dt.$$

Application of the basic filtering theorem from Appendix B to the semimartingale Q_t yields

$$d(\widehat{e^{iux_t}}) = iua\ \widehat{x_t e^{iux_t}}\ dt + \lambda_t E(e^{iu\alpha} - 1)\ e^{iu\hat{x}_t}\ dt$$
$$+ (\widehat{x_t e^{iux_t}} - \hat{x}_t\ \widehat{e^{iux_t}})\ cr^{-1}[dy - c\hat{x}dt]. \qquad 7$$

From the definition of the conditional characteristic function

$$\phi_t(u) \stackrel{\Delta}{=} \widehat{e^{iux_t}} \qquad 8$$

and substitution of this equation into equation (7) results in

$$d\phi_t(u) = au\phi_t'(u)dt + \lambda_t\phi_t(u)(e^{iu\alpha} - 1)dt$$
$$- [i\phi_t'(u) + \hat{x}_t\phi_t(u)]cr^{-1}\ [dy_t - c\hat{x}_t dt] \qquad 9$$

where the prime denotes the partial derivative of $\phi_t(u)$ with respect to u. Equation (9) is the desired equation.

Quasimoment Filtering Equations

At this point, a departure from Kwakernaak (1975) is made. The departure is from the use of cumulants or central moments to quasimoment functions used to define the characteristic function.

Quasimoment Functions

From Appendix A it was seen that the characteristic function $\phi_t(u)$ can be expressed as

$$\phi_t(u) = e^{iu\hat{x}_t - \frac{1}{2}uP_t u} \{1 + \sum_{n=3}^{\infty} \frac{(iu)^n}{n!} k_n\} \qquad 10$$

where the k_n's are the quasimoment functions. In Appendix A it was seen that the first and second quasimoments were zero. From equation (10) it is seen that if all of the k's are zero, then the characteristic function describes a Gaussian process with mean $\hat{x}_t$ and variance P_t. By using this expression for the characteristic function, the deviation from or approach to a Gaussian process can be readily obtained.

Filtering Equations

To obtain the optimal filtering equations, the characteristic function in equation (10) is substituted into equation (9). Using the differentiation rule with equation (10), collecting terms in like powers of u and dropping terms of o(dt) the derivation is completed. This process is followed below for each term in equation (9).

First, the characteristic function from equation (10) is

$$\phi_t(u) = e^{iu\hat{x}_t - \frac{1}{2}uP_t u} [1 - i \frac{u^3}{3!} k_3 + \frac{u^4}{4!} k_4 + i \frac{u^5}{5!} \ . \ . \ .]$$

so that an expansion of the exponential function yields

$$\begin{aligned} \phi_t(u) = {} & 1 + iu\hat{x} - \frac{u^2}{2} (P + \hat{x}^2) - i \frac{u^3}{3!} (3P\hat{x} + \hat{x}^3 + k_3) \\ & + \frac{u^4}{4!} (\hat{x}^4 + 6P\hat{x}^2 + 3P^2 + 4k_3\hat{x} + k_4) \qquad 11 \\ & + i \frac{u^5}{5!} [\hat{x}^5 + \frac{15}{2} P\hat{x}^3 + 15P^2\hat{x} + 10k_3(P + \hat{x}^2) + 5k_4\hat{x} + k_5] + O(u^6). \end{aligned}$$

Taking the partial derivative with respect to u of $\phi_t(u)$ in equation (11), the first term on the right hand side of equation (9), $au \frac{\partial\phi}{\partial u} dt$, becomes

$$au \frac{\partial\phi}{\partial u} dt = [iua\hat{x} - \frac{u^2}{2} (2aP + 2a\hat{x}^2) - \frac{iu^3}{3!} (3aP\hat{x} + 3a\hat{x}^3 + 3ak_3)$$
$$+ \frac{u^4}{4!} (4a\hat{x}^4 + 24aP\hat{x}^2 + 12aP^2 + 16ak_3\hat{x} + 4ak_4) \qquad 12$$
$$+ \frac{iu^5}{5!} (5ak_5 + . . .) + . . .]dt.$$

The second term in equation (9) is obtained by expanding the characteristic function $e^{iu\alpha}$ in terms of its moments α_i obtaining

$$\lambda\phi(e^{iu\alpha} - 1)dt = \lambda\{iu\alpha_1 - \frac{u^2}{2!}(\alpha_2 + \alpha_1\hat{x}) - i \frac{u^3}{3!}(\alpha_3 + 3\alpha_2\hat{x} + P + \hat{x}^2) \qquad 13$$
$$+ \frac{u^4}{4!}[\alpha_4 + 4\alpha_3\hat{x} + 12\alpha_2(P + \hat{x}^2) + 3P + \hat{x}^3] + . . . dt.$$

The last term in equation (9) is obtained in a similar fashion and becomes

$$[i \frac{\partial\phi}{\partial u} + \hat{x}\phi(u)]cr^{-1} [dy - c\hat{x}dt] =$$
$$[iuP - \frac{u^2}{2!} (k_3 + 2P\hat{x}) - i \frac{u^3}{3!}(k_4 + 3P\hat{x}^2 + 3k_3\hat{x} + cP^2) \qquad 14$$
$$+ \frac{u^4}{4!} (k_5 + . . .] cr^{-1}[dy - c\hat{x}dt].$$

Now the left hand side of equation (9) will be evaluated. Expansion of $d\phi_t$ yields

$$d\phi = iud(\hat{x}) - \frac{u^2}{2}(d(P) + d(\hat{x}^2)) - i\,\frac{u^3}{3!}(3d(P\hat{x}) + d(\hat{x}^3) + d(k_3)$$

$$+ \ldots \qquad 15$$

Collection of like powers of "u" in equations (15), (12), (13), and (14) yields for iu

$$d\hat{x} = a\hat{x}dt + Pcr^{-1}\,(dy - c\hat{x}dt) + \lambda\alpha_1 dt. \qquad 16$$

For $\frac{u^2}{2!}$, collection of terms of o(dt) and recognition that

$$d\hat{x}d\hat{x} = Pcr^{-1}cPdt + o(dt) \qquad 17$$

yields

$$dP = 2aPdt + k_3cr^{-1}(dy - c\hat{x}dt) - Pcr^{-1}cPdt + \lambda\alpha_2 dt. \qquad 18$$

For $i\,\frac{u^3}{3!}$, collection of terms of o(dt) and recognition that

$$dPd\hat{x} = k_3Pcr^{-1}cdt + o(dt) \qquad 19$$

yields

$$dk_3 = 3ak_3dt + k_4cr^{-1}(dy - c\hat{x}dt) - dk_3Pcr^{-1}\,cdt + \lambda\alpha_3 dt. \qquad 20$$

For $\frac{u^4}{4!}$, a similar set of steps yields

$$dk_4 = 4ak_4dt + k_5cr^{-1}(dy - c\hat{x}dt) - 4k_4Pcr^{-1}cdt - 3k_3^2cr^{-1}cdt$$

$$+ \lambda\alpha_4 dt \qquad 21$$

where

$$dk_3 d\hat{x} = k_4 Pcr^{-1} cdt + o(dt) \tag{22}$$

and

$$dPdP = k_3^2 cr^{-1} cdt + o(dt). \tag{23}$$

At this point, equations (17), (18), (20) and (21) are the scalar versions of the results of Fisher (1967). This alternate approach in the filtering equation derivation confirms their correctness. The results obtained by Eckberg (1970) differ in that starting with the first of these filtering equations, $n = 1$, an extra term, $\sim k_{n+2}$ appears in the n^{th} equation. The difference between Eckberg's equation and those derived here stems from Eckberg's use in the start of his derivations of a conditional probability density function that is questionable. The results in equations (17), (19), (22 and (23) are obtained from the relationship $dydy \sim 0(dt)$.

Continuation of the process for k_5 and k_6 yields

$$\begin{aligned} dk_5 &= 5ak_5 dt + (k_6 - 10k_3^2)cr^{-1}(dy - c\hat{x}dt) \\ &\quad - 5k_5 Pcr^{-1}cdt - \frac{15}{2} k_4 Pcr^{-1}cdt + \lambda\alpha_5 dt \end{aligned} \tag{24}$$

$$\begin{aligned} dk_6 &= 6ak_6 dt + (k_7 - 15k_4)cr^{-1}(dy - c\hat{x}dt) \\ &\quad - 6k_6 Pcr^{-1}cdt - 3k_5 k_3 cr^{-1}cdt - \frac{15}{2} k_4^2 cr^{-1}cdt \\ &\quad + \lambda(\alpha_6 + 20\alpha_3 k_3)dt. \end{aligned} \tag{25}$$

For applications other than a scalar problem, the sixth quasimoment equations would result in considerable complexity for implementation. The first moment is a vector, the second a matrix, the third is a third order tensor, etc. Consequently, the process of deriving the optimal filtering equations will be terminated at this point.

Summary

In this chapter, the optimal filtering equations have been obtained up to the sixth quasimoment equation via quasimoment functions. It can be seen from these equations that if the quasimoments are zero, the usual linear filter is obtained. The non-linear equations are used in the next section to obtain approximate solutions by truncation of this set at arbitrary points. Up to the fourth quasimoment, the results obtained are the same as the scalar version of those obtained by Fisher (1967) using an alternate approach.

CHAPTER THREE
PERFORMANCE OF THE FILTERING EQUATIONS

Introduction

In this chapter, the performance of the truncated filtering equations will be addressed. First, using the conditional characteristic equation, it will be shown that only for the case of systems driven by continuous Gaussian white noise will the filtering equations become linear. This represents a truncation of the optimal filtering equations at the second moment equation. It will be shown that only under limiting conditions of large jump rate and small jump amplitudes will this be a condition for the optimal filtering equations to converge to the linear filter.

Next, the performance of linear filtering equations is examined using a Lyapunov function. It is shown that the performance of the linear filter depends on a combination of response and stability.

The non-linear filtering equation's performance will be examined under two conditions, low and high signal-to-noise, representing combinations of the system parameters. For the low signal-to-noise condition, use is made of the stability theorem of Perron (e.g., Bucy and Joseph, 1968) to demonstrate stability, and given stability, with stationary values of the quasimoments, demonstrate the expected poor performance of the truncated filtering equations and the decoupling of the higher moments. For the high signal-to-noise condition, use is made of a Lyapunov function constructed by Krasovskii's technique

(e.g., Ogata, 1967). Using this technique, conditions for truncated filter stability are established. Couplings between the higher moments of the filtering equations are shown to be significant using the values of the stationary quasimoments.

As an outcome of the Lyapunov stability analysis of the truncated non-linear filtering equations, two conditions are established for filter stability. The first, that the system, equation (1), be stable, and the second that the second moment, the estimation error covariance, be positive semi-definite are required for filter stability. In the next chapter, numerical examples will be presented to demonstrate filter performance with and without the imposition of the second of these two constraints.

Convergence to the Optimal Linear Filter -- The General Case

Here, the conditions for the convergence to an optimal linear filter for the general case will be examined. Following the idea of Eckberg (1970), the conditional characteristic function for the estimation error will be used to show that only when the characteristic function of the driving noise is an even function of the dummy variable "u" and is quadratic will the optimal filter be linear with a deterministic second moment (estimation error variance).

The characteristic function for the compound Poisson process is an even function of "u" when the jump amplitude is symmetrically disturbed with zero mean. It will be shown that only when the jump amplitudes become infinitesimal with large jump rates will an optimal linear filter be approached.

Conditional Characteristic Equation for the Estimation Error

The conditional characteristic equation for the estimate, equation (9), is rewritten here as

$$d\phi_x = ua \frac{\partial \phi_x}{\partial u} dt + \text{Even}(u)dt - [i \frac{\partial \phi_x}{\partial u} + \hat{x}\phi_x]cr^{-1} [dy - c\hat{x}dt] \qquad 26$$

where the subscript denotes the characteristic function for the estimate, $\hat{x}_t$. The characteristic function for the driving noise, $\lambda_t \phi_t(u)(e^{iu\alpha} - 1)$, for the compound Poisson process has been replaced by Even(u) to denote an even function of "u." The estimation error is defined as

$$e = x - \hat{x}. \qquad 27$$

The characteristic function for the error then becomes

$$\phi_e = e^{-iu\hat{x}} \phi_x. \qquad 28$$

Using this relationship, the following equations result:

$$d\phi_x = e^{iu\hat{x}}[iud\hat{x}\phi_e + d\phi_e] \qquad 29$$

and

$$\frac{\partial \phi_x}{\partial u} = e^{iu\hat{x}}[i\hat{x}\phi_e + \frac{\partial \phi_e}{\partial u}]. \qquad 30$$

The substitution of equations (29) and (30) into equation (26) yields

$$d\phi_e = us \frac{\partial \phi_e}{\partial u} dt + \text{Even}(u)dt - i \frac{\partial \phi_e}{\partial u} cr^{-1}[dy - c\hat{x}dt]$$

$$- iud\hat{x}\phi_e + iua\hat{x}\phi_e dt. \qquad 31$$

This is the equation of the conditional characteristic function for the estimation error given by equation (27).

Solution for Continuous Gaussian Driving Noise

For continuous Gaussian driving noise in equation (1), from Jazwinski (1970), the characteristic function ϕ_e is

$$\phi_e = e^{-\frac{1}{2}uPu}. \tag{32}$$

The characteristic function for the driving noise is

$$\text{Even}(u) = -\tfrac{1}{2}\phi_e uqu \tag{33}$$

where $E(d\sigma d\sigma) = qdt$. From equation (32) the following symmetry properties hold for the function $\phi_e'(u)$ defined as $\phi_e'(u) \triangleq \phi_e(-u)$;

$$\phi_e'(u) = \phi_e(-u) \tag{34}$$

$$\phi_e(u) = \phi_e'(-u) \tag{35}$$

$$d\phi_e(u) = d\phi_e'(-u) \tag{36}$$

$$\frac{\partial\phi_e(u)}{\partial u} = -\frac{\partial\phi_e'(-u)}{\partial u}. \tag{37}$$

Using these relationships, an equation for $\phi_e'(-u)$ can be obtained. This equation is

$$d\phi_e'(-u) = ua\frac{\partial\phi_e'(-u)}{\partial u}dt - \tfrac{1}{2}uqu\,\phi_e'(-u)dt + i\frac{\partial\phi_e'(-u)}{\partial u}cr^{-1}(dy - c\hat{x}dt)$$

$$+ iud\hat{x}\phi_e'(-u) - iua\hat{x}\,\phi_e'(-u)dt. \tag{38}$$

Substituting equation (33) into equation (31), substracting equation (38) from equation (31) and using equation (36) yields

$$0 = \frac{\partial \phi_e(u)}{\partial u} cr^{-1} (dy - c\hat{x}dt) + ud\hat{x}\, \phi_e(u) - ua\hat{x}dt\, \phi_e(u). \tag{39}$$

This equation is a differential equation for $\phi_e(u)$. The solution of equation (39) is

$$\phi_e(u) = \exp(-\tfrac{1}{2}u \frac{d\hat{x} - a\hat{x}dt}{cr^{-1}(dy - cxdt)} u). \tag{40}$$

From equations (40) and (32) the following is obtained:

$$d\hat{x} = a\hat{x}dt + Pcr^{-1}(dy - c\hat{x}dt). \tag{41}$$

This equation is identical to equation (16) when $\alpha_1 = 0$. Instead of subtracting to obtain equation (39), adding the equations yields

$$d\phi_e(u) = 2au \frac{\partial \phi_e(u)}{\partial u} dt - 2 \cdot \tfrac{1}{2}uqu\, \phi_e(u)dt. \tag{42}$$

Using the differentiation rule, equation (4), on $\phi_e(u)$ defined by equation (32) yields

$$d\phi_e(u) = (-dP - Pcr^{-1}cPdt)u^2\, e^{\frac{1}{2}uPu} + o(dt). \tag{43}$$

From equations (43) and (42), the estimation error covariance is

$$dP = 2aPdt + qdt - Pcr^{-1}cPdt. \tag{44}$$

Therefore, for the case of continuous Gaussian driving noise, a linear filter with deterministic error variance is obtained. That is, the error variance is not coupled with the state estimate, $\hat{x}_t$.

Compound Poisson Process Driving Noise Approach to Gaussian

For this case, the function Even(u) can be written as

$$\text{Even}(u) = [-\frac{u^2}{2!}\lambda\alpha_2 + \frac{u^4}{4!}\lambda\alpha_4 + \ldots]\phi_e(u) \qquad 45$$

Since this function is even, the process used to obtain equation (40) will also produce the same equation as equation (40). However, picking the process up at equation (42), higher powers of "u" enter the equations as a result of equation (45). If all terms above the quadratic in equation (45) were zero, i.e., $\lambda\alpha_4 \to 0$, then the above process will again yield a linear filter with deterministic second moment. If this were not true, then equation (32) would have to be amended to provide higher powers of "u"; therefore partial derivatives in equation (42) would produce additional terms as in the application of the differentiation rule.

In general, the optimal filter becomes linear when all terms $\lambda\alpha_4$, $\lambda\alpha_6$, . . . become zero. It has been shown (e.g., Laning and Battin, 1956 and Snyder, 1975) that a compound Poisson process tends to a continuous Gaussian process in the limit as the Poisson rate parameter $\lambda \to \infty$ and the amplitude variance, $\alpha_2 \to 0$ while the product $\lambda\alpha_2$ is a constant. If the amplitude distribution for the jumps is such that higher moments are related to lower moments by a power, i.e., $\alpha_4 \sim \alpha_2^n$, $n > 1$, then the above limiting process will then yield a linear optimal filter. This case will be examined later using stationary quasimoments, where the same result will be obtained.

From these arguments, only in the limits $\lambda \to \infty$, $\alpha_2 \to 0$ as $\lambda\alpha_2$ = constant will the optimal filter become linear. Other conditions,

based on the system parameters a, λ, α_2, c and r, will be established to determine when the truncated filtering equations' performance yields improved performance compared with the linear filter.

Linear Filter Performance

The linear filter performance can be assessed by examining the rate of convergence of the Lyapunov function

$$V(\hat{x}) = \hat{x}P^{-1}\hat{x} \tag{46}$$

where $\hat{x}$ satisfies

$$d\hat{x}/dt = (a - Pcr^{-1}c)\hat{x} \tag{47}$$

and P is given by the variance equation

$$dP/dt = 2aP + \lambda\alpha - Pcr^{-1}cP. \tag{48}$$

Filter stability is assured if $\dot{V}(\hat{x})$ is strictly negative. Therefore,

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \hat{x}}\frac{d\hat{x}}{dt}$$

$$= - [\frac{dP/dt}{P} - 2\,\frac{d\hat{x}/dt}{\hat{x}}]V \tag{49}$$

$$= - [(2a + \frac{\lambda\alpha}{P} - Pcr^{-1}c) - 2(a - Pcr^{-1}c)]V$$

$$\therefore\ \dot{V} = -[\frac{\lambda\alpha}{P} + Pcr^{-1}c]V \tag{50}$$

implies that the larger $\frac{\lambda\alpha}{P}$ and $Pcr^{-1}c$ then the more rapid the

convergence of the linear filter to its estimate. For the scalar constant system that is stable, the stationary value of P is given as

$$\bar{P} = \frac{-ar}{c^2}\left(\sqrt{1 + \frac{\lambda\alpha c^2}{a^2 r}} - 1\right). \qquad 51$$

Therefore, these two quantities can be expressed as

$$\frac{\lambda\alpha}{\bar{P}} = \frac{-\frac{\lambda\alpha c^2}{ar}}{\left(\sqrt{1 + \frac{\lambda\alpha c^2}{a^2 r}} - 1\right)} \qquad 52$$

$$\frac{\bar{P}c^2}{r} = -a\left(\sqrt{1 + \frac{\lambda\alpha c^2}{a^2 r}} - 1\right) \qquad 53$$

Improved linear filter performance is obtained when these two quantities are large.

First, examine the stationary signal-to-noise ratio $\frac{\lambda\alpha c^2}{2ar}$, when $\frac{\lambda\alpha c^2}{ar} >> 1$. This limit yields the following relationships:

$$\frac{\lambda\alpha}{\bar{P}} \sim c\sqrt{\frac{\lambda\alpha}{r}} \qquad 54$$

$$\frac{\bar{P}c^2}{r} \sim c\sqrt{\frac{\lambda\alpha}{r}}\,. \qquad 55$$

These two quantities now become independent of the system parameter "a." Also, the larger the quantity $\sqrt{\frac{\lambda\alpha}{r}}$, then the more rapidly converging and stable the linear filter is.

When $\frac{\lambda\alpha c^2}{ar} << 1$, these two quantities can be approximated as

$$\frac{\lambda\alpha}{\bar{P}} \sim -2a \qquad 56$$

$$\frac{\bar{P}c^2}{r} \sim -\frac{\lambda\alpha c^2}{2ar} \qquad 57$$

The latter of these two is the signal-to-noise ratio. If the condition $\frac{\lambda\alpha c^2}{ar} << 1$ is satisfied, then $\frac{\bar{P}c^2}{r}$ is also small. Therefore, the convergence depends on the system parameter "a." Reexamining the variance for this condition yields

$$\bar{P} \simeq -\frac{\lambda\alpha}{2a} \qquad 58$$

which is just the process noise. Under this condition even though the filter is stable, there is little improvement over the process uncertainty by using filtering.

In summary, the optimal linear filter performance improves when $\frac{\lambda\alpha c^2}{a^2 r} >> 1$ or $c\sqrt{\frac{\lambda\alpha}{r}} > a$. The filter is stable but yields poor performance when $\frac{\lambda\alpha c^2}{ar} << 1$.

Performance of the Truncated Non-linear Filter

Here, the stability and asymptotic performance of the truncated non-linear filtering equations will be discussed. Using the previous results for the linear filter as motivation, the filter stability and asymptotic performance for low signal-to-noise will be addressed. This condition will result in negligible coupling between the higher moments

of the filtering equations. Then the stability of the truncated non-linear filtering equations will be addressed. Two conditions for their stability will be established. Assuming compliance with these two conditions assuring the existence of stationary values for the quasi-moments, non-negligible couplings will be shown to exist between the truncated non-linear filtering equations for large signal-to-noise conditions.

Stability of the Non-linear Filtering Equations

The non-linear filter performance is comprised of responsiveness and stability as in the case of the linear filter. To evaluate the filter's stability, it is instructive to rewrite equations (16), (18), (20), (24) and (25) as a non-linear vector stochastic differential equation as shown in Table 1. It is seen from this table that the equations are coupled and exhibit different characteristics for each additional pair of moments included in the state vector i.e., $\hat{x}$ and P; $\hat{x}$, P, k_3 and k_3 etc. Up to the fifth quasimoment term, the innovations process $(dy - c\hat{x}dt)$ enters into a moment equation by the next higher moment. Rewriting the equation in Table 1 as shown in Table 2 allows the view of these equations as those comprised of a "linear" term, $A\underline{\hat{x}}$, and non-linear coupled terms. This can also be illustrated by the following vector stochastic equation:

$$d\underline{x} = A\underline{x}dt + \underline{h}(\underline{x}, t)\, dt + \underline{r}(\underline{x}, t)\, \frac{c}{r}\, dy \qquad 59$$

If A were a stable constant matrix, then use of Perron's theorem (e.g., Bucy and Joseph, 1968) can be made to evaluate the system

TABLE 1

NONLINEAR FILTERING EQUATIONS UP TO THE SIXTH QUASIMOMENT

$$
\underbrace{\begin{bmatrix} d\hat{x} \\ dP \\ dk_3 \\ dk_4 \\ dk_5 \\ dk_6 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}}_{d\underline{x}}
=
\underbrace{\begin{bmatrix}
a & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2a - \frac{Pc^2}{r} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 3(a - \frac{Pc^2}{r}) & 0 & 0 & 0 & \cdots \\
0 & 0 & -3\,\frac{k_3c^2}{r} & 4(a - \frac{Pc^2}{r}) & 0 & 0 & \cdots \\
0 & 0 & -\frac{15}{2}\,\frac{k_4c^2}{r} & 0 & 5(a - \frac{Pc^2}{r}) & 0 & \cdots \\
0 & 0 & -3\,\frac{k_5c^2}{r} & -\frac{15}{2}\,\frac{k_4c^2}{r} & 0 & 6(a - \frac{Pc^2}{r}) & \cdots \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & &
\end{bmatrix}
\begin{bmatrix} \hat{x} \\ P \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} dt}_{\underline{f}(\underline{x},\, t)dt}
+
\underbrace{\begin{bmatrix} \alpha_1\lambda \\ \alpha_2\lambda \\ \alpha_3\lambda \\ \alpha_4\lambda \\ \alpha_5\lambda \\ \alpha_6\lambda \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} dt}_{+\ \underline{g}dt}
$$

$$
+ \frac{c}{r}(dy - c\hat{x}dt)
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & -10k_3 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & -15 & 0 & 0 & 1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{bmatrix}
\begin{bmatrix} \hat{x} \\ P \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}
$$

$$
+ \frac{c}{r}(dy - c\hat{x}dt)\ \underline{q}(\underline{x},\, t)
$$

TABLE 2

REFORMULATED NONLINEAR FILTERING EQUATIONS FROM TABLE 1

$$\underset{d\underline{x}}{\begin{bmatrix} d\hat{x} \\ dP \\ dk_3 \\ dk_4 \\ dk_5 \\ dk_6 \end{bmatrix}} = \underset{A\underline{x}dt}{\begin{bmatrix} a - \frac{Pc^2}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a - \frac{Pc^2}{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3(a - \frac{Pc^2}{r}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 4(a - \frac{Pc^2}{r}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 5(a - \frac{Pc^2}{r}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 6(a - \frac{Pc^2}{r}) \end{bmatrix} \begin{bmatrix} \hat{x} \\ P \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix} dt} + \begin{bmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \\ \lambda\alpha_3 \\ \lambda\alpha_4 \\ \lambda\alpha_5 \\ \lambda\alpha_6 \end{bmatrix} dt$$

$$+ \underset{+\,\underline{h}(\underline{x},\ t)dt}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\hat{x}c^2}{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hat{x}c^2}{r} & 0 & 0 & 0 \\ 0 & 0 & -3\frac{k_3c^2}{r} & 0 & -\frac{\hat{x}c^2}{r} & 0 & 0 \\ 0 & +10\frac{k_3\hat{x}c^2}{r} & -\frac{15}{2}\frac{k_4c^2}{r} & 0 & 0 & -\frac{\hat{x}c^2}{r} & 0 \\ 0 & +15\frac{k_4\hat{x}c^2}{r} & -3\frac{k_5c^2}{r} & -\frac{15}{2}\frac{k_4c^2}{r} & 0 & 0 & -\frac{\hat{x}c^2}{r} \end{bmatrix} \begin{bmatrix} \hat{x} \\ P \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix}} dt +$$

$$+ \underset{+\qquad \underline{r}(\underline{x},\ t)\frac{c}{r}\,dy}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -10k_3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -15 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ P \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix} dy\frac{c}{r}}$$

stability. The matrix A is stable when $(a - \frac{Pc^2}{r}) < 0$ and $(2a - \frac{Pc^2}{r}) < 0$. The first of these will reappear later for non-linear filtering. Examining the vector $\underline{h}(\underline{x}, t)$ term by term, it is seen that if products of the first moment, x, and the quasimoments are small, such that $\underline{h}(\underline{x}, t)$ is of order $o(|| x ||)$, then the equation (59) is asymptotically stable. Under these conditions, stationarity of the quasimoments is assured. This low degree of coupling corresponds to low signal-to-noise or a near Gaussian process in the limit discussed earlier. This will be demonstrated later.

If the couplings between the equations are not of order $o(|| x ||)$, then Lyapunov techniques are required to determine the conditions for stability.

The equations in Table 1 will be examined to evaluate their stability when the non-linear terms are not negligible. The stability shall be evaluated using Krasovskii's Theorem (e.g., Bucy and Joseph, 1968). For a general non-linear system of dimension n, the system is described by the following vector differential equation.

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 60$$

Let the Jacobian of $\underline{f}(\underline{x})$ be given by $F(\underline{x})$. Krasovskii's Theorem states that if $\underline{f}(\underline{0}) = \underline{0}$ and $\hat{F}(\underline{x})$ is negative definite where $\hat{F}(\underline{x})$ is defined as

$$\hat{F}(\underline{x}) = F^T(\underline{x}) + F(x) \qquad 61$$

where superscript "T" denotes transpose, then the equilibrium state

$\underline{x} = \underline{0}$ is asymptotically stable. The Lyapunov function is given as

$$V(\underline{x}) = \underline{f}^T(\underline{x})\,\underline{f}(\underline{x}) \qquad 62$$

with

$$\dot{V}(\underline{x}) = \underline{f}^T(\underline{x})\,\hat{F}(\underline{x})\,\underline{f}(\underline{x}). \qquad 63$$

Additionally, if $V(\underline{x})$ tends to infinity as $||\,\underline{x}\,|| \to \infty$, then the equilibrium state is asymptotically stable in the large.

The system of equations to be evaluated with this Lyapunov function construction technique must have zero as their equilibrium state. For the first three, up to the third quasimoment, the unforced equations are written as

$$\begin{aligned} d\hat{x}/dt &= a\hat{x} - \frac{Pc^2}{r}\hat{x} \\ &= (a - \frac{Pc^2}{r})\,\hat{x} \end{aligned} \qquad 64$$

$$\begin{aligned} dP/dt &= 2aP - \frac{P^2c^2}{r} - \frac{k_3\hat{x}c^2}{r} + \lambda\alpha_2 \\ &= (2a - \frac{Pc^2}{r})\,P - \frac{k_3\hat{x}c^2}{r} + \lambda\alpha_2 \end{aligned} \qquad 65$$

$$\begin{aligned} dk_3/dt &= 3ak_3 - \frac{3k_3Pc^2}{r} - \frac{k_4\hat{x}c^2}{r} \\ &= 3(a - \frac{Pc^2}{r})\,k_3 - \frac{k_4\hat{x}c^2}{r} \end{aligned} \qquad 66$$

where the zero mean property of the compound Poisson process has been used. For these equations, P and k_4 have nonzero equilibrium conditions.

For P, with the equilibrium value associated with its positive semi-definite value is given as

$$P_{ss} = -\frac{ar}{c^2}\left(\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}} - 1\right) \qquad 67$$

and defining

$$\tilde{P} \triangleq P - P_{ss} \qquad 68$$

the desired equation is

$$\dot{\tilde{p}} = 2\tilde{a}\tilde{P} - \frac{\tilde{P}^2 c^2}{r} - \frac{k_3 \hat{x} c^2}{r} \qquad 69$$

where

$$\tilde{a} = a\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}} \qquad 70$$

Below are the resulting equations for zero equilibrium conditions.

$$d\hat{x}/dt = \left(a - \frac{Pc^2}{r}\right)\hat{x} \qquad 71$$

$$d\tilde{P}/dt = 2\tilde{a}\tilde{P} - \frac{\tilde{P}^2 c^2}{r} - \frac{k_3 \hat{x} c^2}{r} \qquad 72$$

$$dk_3/dt = 3\left(\tilde{a} - \frac{\tilde{P}c^2}{r}\right)k_3 - \frac{k_4 \hat{x} c^2}{r} \qquad 73$$

The Jacobian, $F(\underline{x})$, for equations (71) through (73) is

$$F(\underline{x}) = \begin{bmatrix} (a - \frac{Pc^2}{r}) & -\frac{\hat{x}c^2}{r} & 0 \\ -\frac{k_3c^2}{r} & 2(\tilde{a} - \frac{\tilde{P}c^2}{r}) & -\frac{\hat{x}c^2}{r} \\ -\frac{k_4c^2}{r} & -\frac{3k_3c^2}{r} & 3(\tilde{a} - \frac{\tilde{P}c^2}{r}) \end{bmatrix} \qquad 74$$

and the matrix $\hat{F}(\underline{x})$ becomes

$$\hat{F}(\underline{x}) = \begin{bmatrix} 2(a - \frac{Pc^2}{r}) & -(\hat{x} + k_3)\frac{c^2}{r} & -\frac{k_4c^2}{r} \\ -(\hat{x} + k_3)\frac{c^2}{r} & 4(\tilde{a} - \frac{\tilde{P}c^2}{r}) & -(\hat{x} + 3k_3)\frac{c^2}{r} \\ -\frac{k_4c^2}{r} & -(\hat{x} + 3k_3)\frac{c^2}{r} & 6(\tilde{a} - \frac{\tilde{P}c^2}{r}) \end{bmatrix} \qquad 75$$

To determine the requirements for the negative definiteness of the matrix, $\hat{F}(\underline{x})$, Sylvester's criteria (e.g., Ogata, 1967) are used. Applying these criteria yield the following inequalities as sufficient conditions for stability:

$$(a - \frac{Pc^2}{r}) < 0 \qquad 76$$

$$8(a - \frac{Pc^2}{r})(\tilde{a} - \frac{\tilde{P}c^2}{r}) + (\hat{x} + k_3)^2 \frac{c^4}{r^2} > 0 \qquad 77$$

$$2\left(a - \frac{Pc^2}{r}\right)\left[24\left(\tilde{a} - \frac{\tilde{P}c^2}{r}\right)^2 - (\hat{x} + 3k_3)^2 \frac{c^4}{r^2}\right]$$

$$- 6\left(\tilde{a} - \frac{\tilde{P}c^2}{r}\right)(\tilde{x} + k_3)^2 \frac{c^4}{r^2} - 2k_4(\hat{x} + k_3)(\hat{x} + 3k_3)\frac{c^6}{r^3}$$

$$- 4\left(\tilde{a} - \frac{\tilde{P}c^2}{r}\right) k_4^2 \frac{c^4}{r^2} < 0. \qquad 78$$

Under equilibrium conditions, the sufficient conditions become

$$\left(a - \frac{Pc^2}{r}\right) < 0 \qquad 79$$

$$8\left(a - \frac{Pc^2}{r}\right)\tilde{a} > 0 \qquad 80$$

$$12\left(a - \frac{Pc^2}{r}\right)^2 < \frac{k_4^2 c^4}{r^2}\,. \qquad 81$$

The first of these is an obvious result. Combining the first and second requires

$$\tilde{a} < 0 \qquad 82$$

which requires, from equation (70), that the system of equation (1) be stable. Returning to equation (76), if the quantity $\frac{Pc^2}{r}$ became negative such that $(a - \frac{Pc^2}{r}) > 0$, then instability would occur. The last of these inequalities establishes an upper limit for second moment, P, thus precludes an arbitrarily large gain to improve the filter response.

In obtaining the above inequalities, the equilibrium value of P, P_{ss}, was chosen as its positive semi-definite value for a stable system. The inequalities (79) through (81) are based on this selection. The first of these inequalities is assured for P positive semi-definite as was the case for the linear filter. If, P is positive semi-definite, a stable filter results.

In the previous developments, stability conditions up to the fourth quasimoment were determined. Stability conditions for the higher moments, however, must also satisfy the conditions for the lower ones since Sylvester's criterion adds additional equations within the determinant which adds additional inequalities to those previously obtained.

For the cases of low and high couplings, stability is assumed. With this assumption, the stationary values of the quasimoments will be evaluated to add further evidence as to the degree of coupling between the filtering equations and the resulting performance improvements of the truncated non-linear filter relative to the linear filter.

The assumption of a Gaussian jump amplitude distribution will be used in the following. This assumption permits the use of specific numbers rather than proportionality constants in obtaining expressions for the stationary quasimoments. The higher moments of the jump amplitude distribution are related by powers of the lower moments, and for the Gaussian distribution are given as $\alpha_4 = 3\alpha_2^2$, $\alpha_6 = 15\alpha_2^3$, etc.

From the non-linear filtering equations, the following expressions are obtained for the stationary quasimoments:

$$\bar{k}_3 = 0 \tag{83}$$

$$\bar{k}_4 = \frac{-\,3/4\lambda\alpha_2^2}{a\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}} \tag{84}$$

$$\bar{k}_5 = \frac{9/8\ \lambda\alpha_2^2\left(\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}} - 1\right)}{a\left(1 + \frac{\lambda\alpha_2 c^2}{a^2 r}\right)} \tag{85}$$

$$\bar{k}_6 = \frac{\frac{15}{6}\lambda\alpha_2^{\ 3}}{a\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}}\left[\frac{9}{32}\lambda\left(\frac{\frac{\lambda\alpha_2 c^2}{a^2 r}}{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}\right) - 1\right] \tag{86}$$

The expression for the stationary second moment is given by equation (51).

Re-examination of Convergence to Linear Filter

Before proceeding with evaluation the performance of the non-linear filtering equations, the case of high jump rate and low jump amplitude will be re-examined. This will be accomplished using the stationary quasimoments from equation (84) through (86).

Case I; $\lambda \to \infty$, $\alpha_2 \to 0$, $\lambda\alpha_2$ = constant.

From these equations, the term $\lambda\alpha_2$ occurs as $\lambda\alpha_2^n$, n = 1, 2, 3, ... Then as $\lambda \to \infty$ and $\alpha_2 \to 0$ while $\lambda\alpha_2$ = constant,

$\lambda\alpha_2^n \to 0$ for $n > 2$ for all the equations. Therefore, the stationary quasimoments tend to zero with the higher moments approaching zero more rapidly in the limit than the lower ones. For a more general jump amplitude distribution than Gaussian, the same result holds as long as $\alpha_{2(\ell+1)}^n \propto \alpha_{22}^n$ when $m > n$ and $\ell = 1, 2, \ldots$.

Non-linear Filter Performance - Low Signal-to-Noise

Again, using the stationary quasimoments, the performance of the non-linear filtering equations are examined for the case of

$$\frac{\lambda\alpha_2 c^2}{a^2 r} << 1.$$

Case II: $\dfrac{\lambda\alpha_2 c^2}{a^2 r} << 1.$

The stationary quasimoments become

$$\bar{k}_4 \to \frac{3}{2}\alpha_2\,\bar{P} \gtrsim 0 \qquad 87$$

$$\bar{k}_5 \to \frac{9}{8}\left(\frac{\lambda\alpha_2 c^2}{a^2 r}\right)\alpha_2\,\bar{P} \gtrsim 0 \qquad 88$$

$$\bar{k}_6 \to \frac{15}{6}\left[\frac{9}{32}\,\frac{\lambda\alpha_2^3}{a}\left(\frac{\lambda\alpha_2 c^2}{a^2 r}\right) - \frac{ar\alpha_2^2}{c^2}\left(\frac{\lambda\alpha_2 c^2}{a^2 r}\right)\right] \gtrsim 0 \qquad 89$$

where the stationary second moment was found earlier. From equation (58), this moment is

$$\bar{P} \to -\frac{\lambda\alpha_2}{2a} \qquad 58$$

which is the process noise variance.

This case is the case considered by Au (1979). As $\lambda \to 0$, the stationary quasimoments tend to zero. Other limits of the system parameters can also yield the same results, therefore, making this single parameter effect nonunique.

From equation (88), it is seen that for $\frac{\lambda\alpha_2 c^2}{a^2 r} \to 0$ but $\bar{P}$ finite, the stationary quasimoment tends to zero. This is the second "break" of the couplings between the non-linear equations, the first being $\bar{k}_3$. Therefore, when $\frac{\lambda\alpha_2 c^2}{a^2 r} \to 0$, the non-linear filtering equations uncouple and the filtering performance offers little improvement over the linear filter which is no better than the process noise.

To illustrate this further, from the equation in Table 1, the coupling occurs as $\frac{k_i c^2}{r}$. Then these terms become

$$\frac{\bar{k}_4 c^2}{r} \to \frac{3}{4}\left(\frac{\lambda\alpha_2 c^2}{ar}\right)\alpha_2 \qquad 90$$

$$\frac{\bar{k}_5 c^2}{r} \to \frac{9}{16}\left(\frac{\lambda\alpha_2 c^2}{a^2 r}\right)\left(\frac{\lambda\alpha_2 c^2}{ar}\right)\alpha_2 \qquad 91$$

etc.

Therefore, if $\frac{\lambda\alpha_2 c^2}{ar} \to 0$, then the stationary quasimoments tend to zero. Again, under these conditions, the non-linear filtering equations become uncoupled as indicated by the values of the stationary quasimoments.

In summary, for this case, the non-linear filtering equations yield stable results; however, the performance offers little improvement over the linear filter which is no better than the process noise.

Non-linear Filter Performance - High Signal-to-Noise

Finally, the conditions in which the non-linear filtering equations offer a performance improvement are examined. This improvement is accompanied by a potential reduction of the filter's stability characteristics, however, for the following discussions, it is assumed that the conditions previously established for stability are satisfied.

Case III: $\dfrac{\lambda\alpha_2 c^2}{a^2 r} >> 1.$

For this case, the stationary quasimoments are

$$\bar{k}_4 \rightarrow \frac{3}{4}\frac{1}{c}\sqrt{\lambda\alpha_2^3 r} \qquad 92$$

$$\bar{k}_5 \rightarrow \frac{9}{8}\frac{1}{c}\sqrt{\lambda\alpha_2^3 r} \qquad 93$$

$$\bar{k}_6 \rightarrow \frac{15}{6}[\frac{9}{32}\lambda + \frac{1}{c}\sqrt{\lambda\alpha_2^5 r}\ ] \qquad 94$$

Or, as before, the following coupling relationships are obtained:

$$\frac{\bar{k}_4 c^2}{r} \rightarrow \frac{3}{4}c\sqrt{\frac{\lambda\alpha_2^3}{r}} \qquad 95$$

$$\frac{\bar{k}_5 c^2}{r} \rightarrow \frac{9}{8}c\sqrt{\frac{\lambda\alpha_2^3}{r}} \qquad 96$$

etc.

For this case, the stationary second moment is given by

$$\bar{P} \rightarrow \frac{1}{c}\sqrt{\lambda\alpha_2 r}\ . \qquad 97$$

Also, the coupling term is given by

$$\frac{\bar{p}c^2}{r} \rightarrow c\sqrt{\frac{\lambda\alpha_2}{r}} \, . \tag{98}$$

From equations (92) through (96), it is seen, based on the magnitudes of the stationary quasimoments, that if the quantity $c\sqrt{\frac{\lambda\alpha_2}{r}}$ is large, strong couplings exist.

Summary

In this section, the stability and performance of the non-linear filtering equations have been addressed for filters truncated at different moments of the infinite dimensional non-linear filtering equations. It was shown that the linear filter, a truncation at the second moment, exhibits stable performance with improved convergence for systems possessing high signal-to-noise ratios, and stable but poor performance improvement over the process noise for systems possessing low signal-to-noise. For the non-linear filtering equations, it was demonstrated that for low signal-to-noise the non-linear filtering equations became uncoupled when $\frac{\lambda\alpha_2 c^2}{a^2 r} \ll 1$. This condition results in little improvement of the non-linear filtering equations over that provided by the linear filter.

For the truncated non-linear filtering equations with high signal-to-noise and when $\frac{\lambda\alpha_2 c^2}{a^2 r} \gg 1$, it was demonstrated that strong couplings exist between the equations. The potential for improvement offered by the non-linear filtering equations is accompanied by the

likelihood of instability. Conditions were established, using Lyapunov techniques, to assure stability of the truncated non-linear filtering equations.

In the next section, numerical examples will be presented for the cases of high and low signal-to-noise and with and without the lower bound condition for stability of the truncated non-linear filter imposed.

CHAPTER FOUR
NUMERICAL EVALUATION

Introduction

The purpose of this chapter is to evaluate the conditions established in the previous chapter using specific numerical examples. Two of these numerical examples are those reported previously by Eckberg (1970) and Kwakernaak (1975) and two additional examples supplement the range of variation of the signal-to-noise ratio to agree with the magnitude variations of this quantity used earlier. These examples will be simulated using filters truncated at the second moment, fourth quasimoment and sixth quasimoment.

In addition to the previous attempts of Eckberg and Kwakernaak of simulating the filtering equations, others have been reported by Clements and Anderson (1973). In this reference, the non-linear filtering equations of Wonham (1965) for the random telegraph wave were simulated. Reported in this reference and in Eckberg and Kwakernaak were stability problems with the numerical simulations.

For a non-linear system driven by a continuous Wiener process, Astrom (1965) used numerical simulation to obtain the steady-state probability density distributions for several numerical examples. Unlike the other reports, Astrom did not report problems with numerical stability. The numerical algorithm used to simulate the filtering equations and the process defined by equations (1) and (3) is similar

to that used by Astrom; however, as in the previous similar simulation attempts, stability problems were encountered.

Numerical Algorithms

Approximations, for implementation onto a digital computer, were used to simulate the process described by equations (1) and (3). These stochastic differential equations can be interpreted as the limit of the following stochastic difference equations:

$$x_{\Delta t}(t_{n+1}) = x_{\Delta t}(t_n) + ax_{\Delta t}(t_n)\Delta t + \sigma(t_n)\delta_{nm} \tag{99}$$

$$dy_{\Delta t} = c\, x_{\Delta t}(t_{n+1})\, \Delta t + v(t_n)\, \sqrt{\Delta t} \tag{100}$$

where δ_{nm} is the Kronecker delta function defined by

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \tag{101}$$

where the subscripts n and m denote the n^{th} time step separated by time step Δt and the m^{th} time step at which a jump should occur. The filtering equations in Table 1 were simulated in a similar fashion. The filtering equations were implemented as the stochastic difference equation

$$\underline{x}(t_{n+1}) = \underline{x}(t_n) + \underline{f}(\underline{x},\, t_n)\Delta t + \underline{q}\Delta t + \frac{c}{r}(dy_{\Delta t} - c\underline{x}(t_n)\Delta t)g(\underline{x},\, t_n) \tag{102}$$

where $\underline{x}$ is the vector composed of the moments included in the truncated filter equations.

The process $v(t_n)$ is generated using the system routine for developing psuedo random numbers uniformly distributed within the interval (0, 1). Approximate normally distributed random numbers were generated by summing twelve uniformly distributed pseudo random numbers. This result, minus the mean and multiplied by the square root of the apriori variance, yielded the sequence $v(t_n)$.

From Astrom it was demonstrated that the solution of difference equations that approximate stochastic differential equations such as equation (102) will converge (in the sense of mean square) to the solution of the stochastic differential equations. Also the distribution law of the sequence of the difference equations will converge to the distribution laws of the process of the differential equation.

Basis of Evaluation

For each of the numerical examples, three filters all operating on the sample path generated by the system of equations (1) and (3) will be evaluated continuously over an interval of the reciprocal of the jump rate λ. The three filters are 1) linear filter (truncated at the second moment), 2) non-linear filter truncated at the fourth quasimoment, and 3) non-linear filter truncated at the sixth quasimoment.

The measure of performance for the filters is the mean square error. For the i^{th} time step and the j^{th} filter, this error is computed recursively by

$$M_{se_{i+1,j}} = M_{se_{i,j}} + \frac{1}{i+1}[(x_i - \hat{x}_{i,j})^2 - M_{se_{i,j}}] \qquad 103$$

This recursive equation for the mean and another presented later for the variance eliminates the necessity for storing the results for later "batch" processing to obtain the statistical data. The mean square error continuously computed over the time interval permits the correlation between the filter performance and those variables that influence the performance, i.e., P.

Also presented for each of the filters and the numerical examples is the continuously computed mean of the second moment and its variance. The recursive equations used to calculate the mean and variance for the i^{th} time and j^{th} filter are given by

$$M_{P_{i+1,j}} = M_{P_{i,j}} + \frac{1}{i+1}(P_{i,j} - M_{P_{i,j}}) \qquad 104$$

$$\sigma^2_{P_{i+1,j}} = \sigma^2_{P_{i,j}} + \frac{1}{i}\left[\frac{i}{i+1}(P_{i,j} - M_{P_{i,j}})^2 - \sigma^2_{P_{i,j}}\right] \qquad 105$$

The filter stability and convergence are influenced by the instantaneous value of P.

The filter performances are presented for a time interval of the inverse of the jump rate λ. During this interval, the probability that no more than one event occurring is 0.736. If an interval twice as long were used,then this probability would drop to 0.405. Corresponding to the assumed Gaussian distributed jump amplitudes, for each of the numerical examples, two amplitudes were examined, one and two sigma values of the amplitude standard deviation.

The computer simulation runs were started with all variables, mean, second and quasimoments, equal to zero. One hundred points were

generated and processed by the filters before the jump in equation (99) occurred.

Discussion of Results

Results from the simulations for the numerical examples are presented to show the relative performances of the filtering equations developed earlier. The unconstrained performance of the fourth and sixth quasimoment filtering equations will be presented. They illustrate the potential for improvement for high amplitude jumps and the potential for momentary instability and associated poor performance for low amplitude jumps. Two of these numerical examples will be used to demonstrate the improvement in performance for both the fourth and sixth quasimoment filtering equations when the second moment is constrained to be positive semi-definite.

This constraint is implemented within the computer simulation programs as a simple test to determine if the current value of the second moment would become less than zero if the incremental value of the second moment, determined by equation (102), were added to it. If this second moment were to become less than zero, then the increment is not added and the second moment retained its value. This process continues each time step until the simple test results in a positive semi-definite second moment. This process is a suboptimal procedure; however, it yields improved performance in the simulation results to be seen later.

The system parameter values assumed for the numerical examples and other combinations of those values are presented in Table 3.

TABLE 3

SYSTEM PARAMETERS AND STATIONARY VALUES FOR THE NUMERICAL EXAMPLES SIMULATED

Parameter	Definition	Eckberg (Example 1)	Kwakernaak (Example 2)	Example 3	Example 4
a	system response	-250	-4	-1	-1
λ	jump rate	250	1	1	1
α_2	jump amplitude variance	67	1	1	1
c	measurement amplification	1	1	1	1
r	measurement error variance	.01	.02	.02	1
$\frac{\lambda\alpha_2 c^2}{2ar}$	signal-to-noise ratio	3350	6.25	25	.5
$\bar{P}$		11.68	.0825	.1228	.4142
$\bar{k}_4$		638.54	.0923	.1050	.5303
$\bar{k}_5$		776915	.0703	.1355	.2330
$\bar{k}_6$		305843097.	3.5038	4.4861	3.8771
$\frac{\lambda\alpha_2 c^2}{a^2 r}$		26.8	3.125	50	1
Δt		1.0 E-8	.2 E-5	1.0 E-5	1.0 E-4
$c\sqrt{\frac{\lambda\alpha_2}{r}}$		1294.2	7.0711	7.0711	1

The first two examples are from the other published simulation studies. The last two were added to increase the range of signal-to-noise conditions. The Kwakernaak example and the third example will be used for comparison when the constrained value for the second moment is used.

Unconstrained Second Moment Comparisons

It is seen from the figures for the mean squared error, the benefit of the non-linear filtering equations over the linear filter is the increased response of the estimate of the state value to the state. The curves illustrating the mean of the second moment reflect the higher initial weighting of the measurements as compared to the linear filter second moment. Through examination of the variance about the mean of the second moment, when this quantity is large relative to the mean, it is seen that the non-linear filter's performance is poor. These trends are consistent for all the examples with high signal-to-noise using unconstrained second moments in the filtering equations. For the example for low signal-to-noise, there is little difference between the filters' performance with sluggish responses for all.

The performances of the linear filter and the fourth quasimoment filter are comparable for jump amplitude at the one sigma level, while the sixth quasimoment filter exhibits relatively poor performance. For jump amplitudes of the two sigma level, the sixth quasimoment filter's performance is better than the fourth quasimoment filter which is better than the linear filter. For the high signal-to-noise examples, the fourth quasimoment filter exhibits the best overall

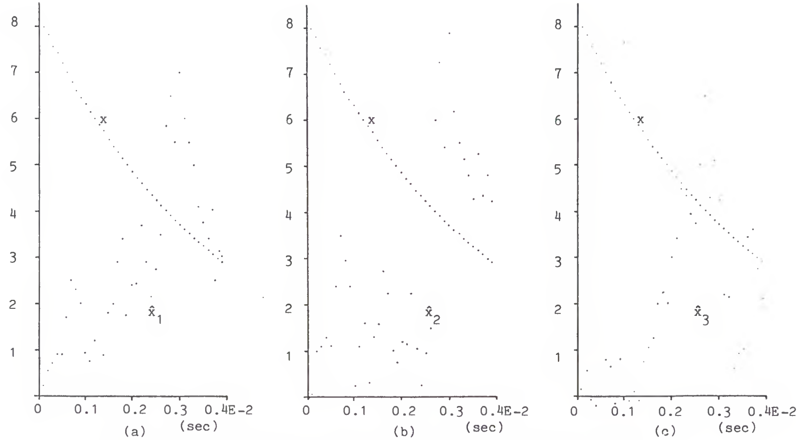

FIGURE 1. Eckberg example, Unconstrained, One sigma jump linear, 4th and 6th quasimoment filter vs state

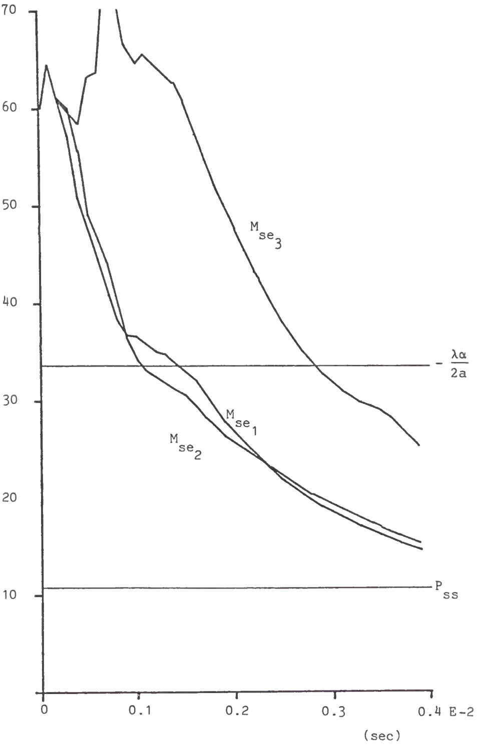

FIGURE 2. Eckberg example, Unconstrained, One sigma jump mean of second moment for non-linear filters

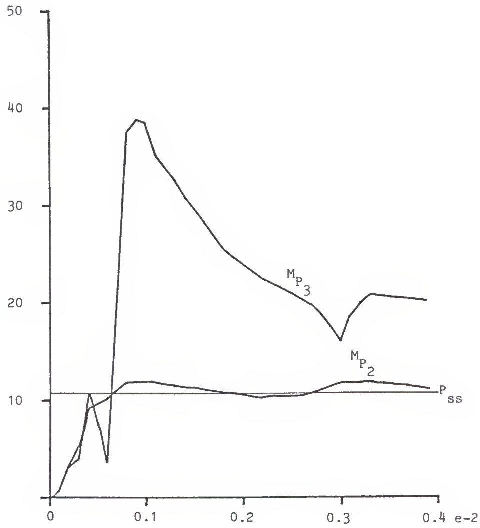

FIGURE 3. Eckberg example, Unconstrained, One Sigma jump mean squared error for all filters

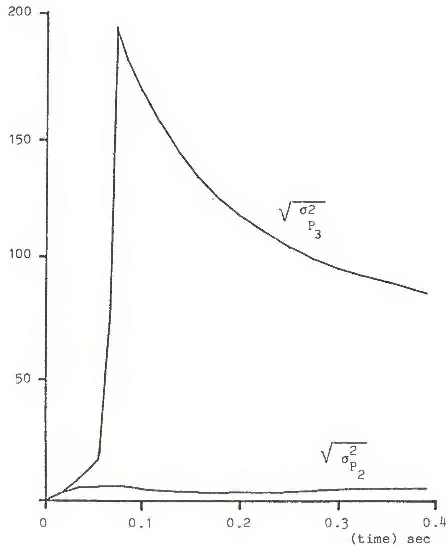

FIGURE 4. Eckberg example, Unconstrained, One sigma jump variance of second moment for non-linear filters

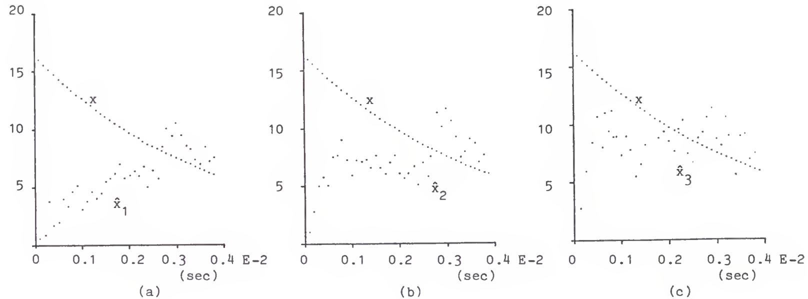

FIGURE 5. Eckberg example, Unconstrained, Two sigma jump linear, 4th and 6th quasimoment filter vs state

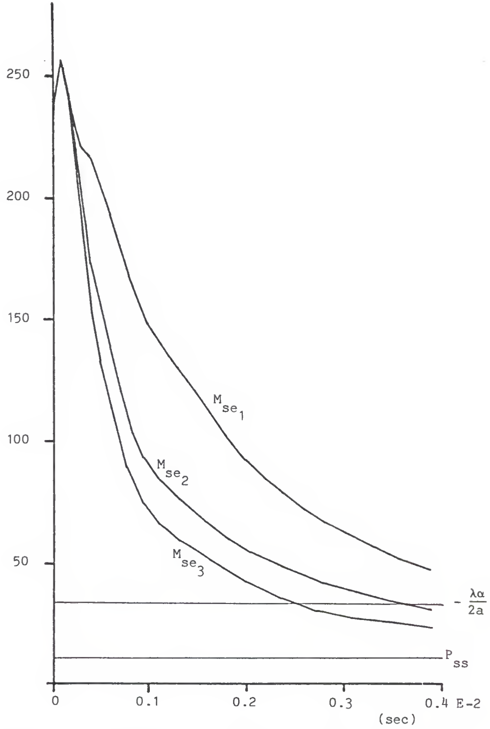

FIGURE 6. Eckberg example, Unconstrained, Two sigma jump mean squared error for all filters

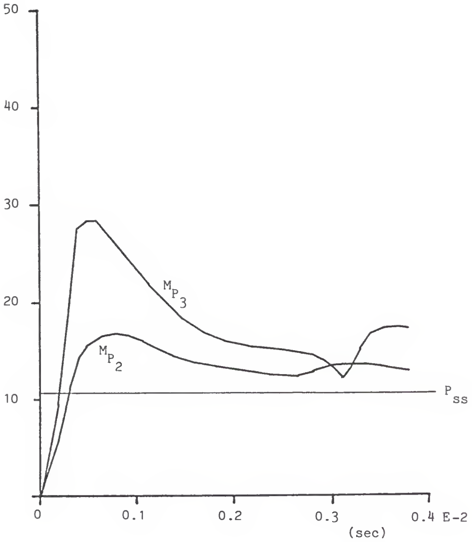

FIGURE 7. Eckberg example, Unconstrained, Two sigma jump mean of second moment for non-linear filters

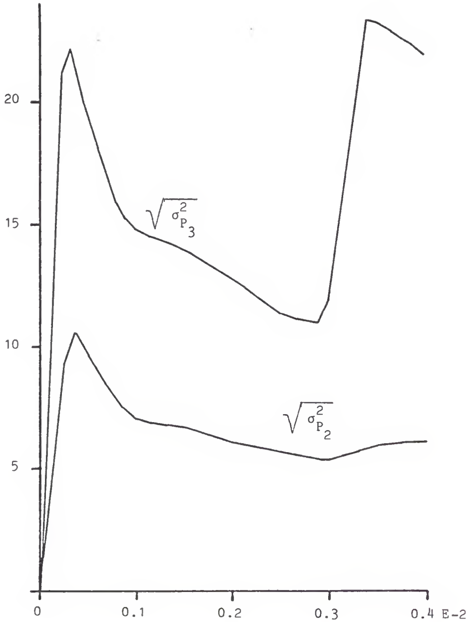

FIGURE 8. Eckberg example, Unconstrained, Two sigma jump variance of second moment for non-linear filters

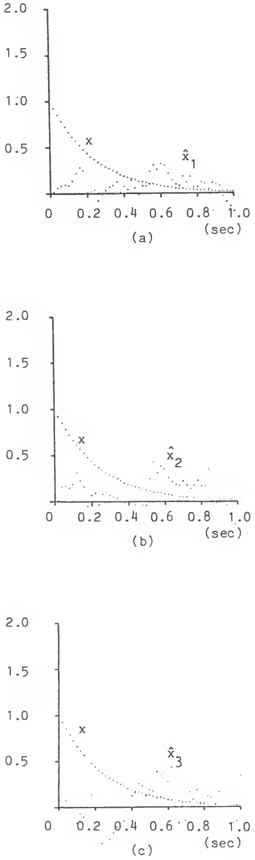

FIGURE 9. Kwakernaak example, Unconstrained, One sigma jump linear, 4th and 6th quasimoment filter vs state

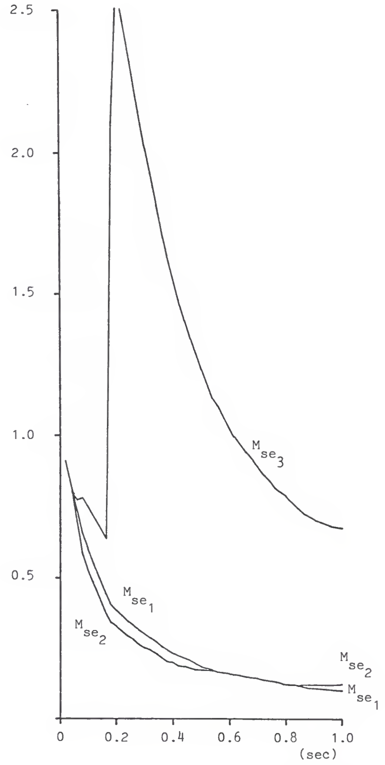

FIGURE 10. Kwakernaak example, Unconstrained, One sigma jump mean squared error for all filters

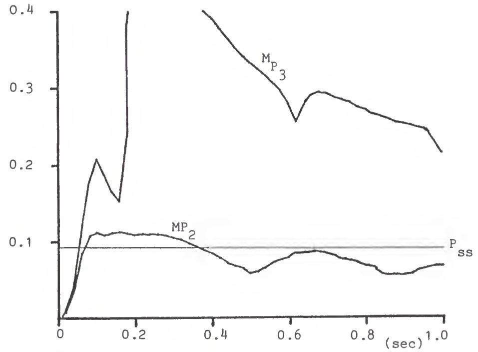

FIGURE 11. Kwakernaak example, Unconstrained, One sigma jump mean of second moment for non-linear filters

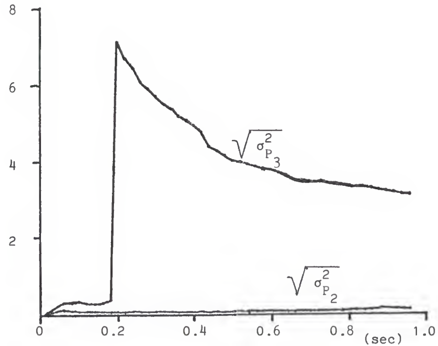

FIGURE 12. Kwakernaak example, Unconstrained, One sigma jump variance of second moment for non-linear filters

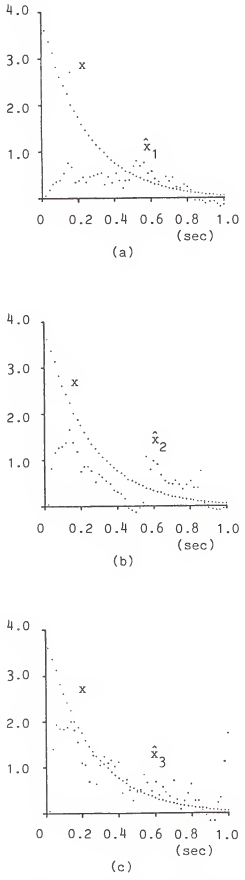

FIGURE 13. Kwakernaak example, Unconstrained, Two sigma jump linear, 4th and 6th quasimoment filter vs state

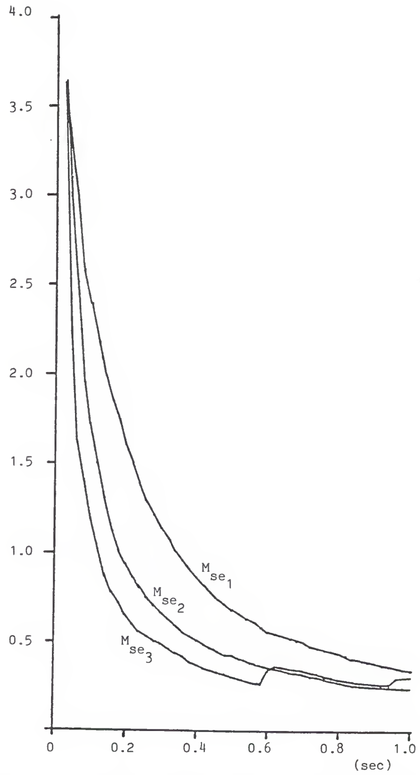

FIGURE 14. Kwakernaak example, Unconstrained, Two sigma jump mean squared error for all filters

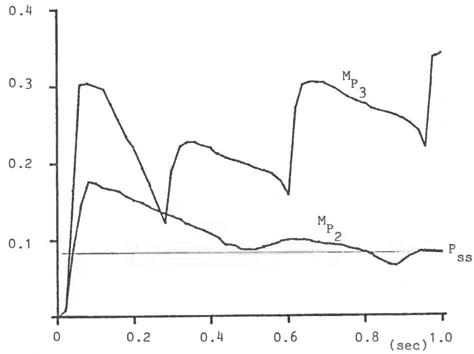

FIGURE 15. Kwakernaak example, Unconstrained, Two sigma jump mean of second moment for non-linear filters

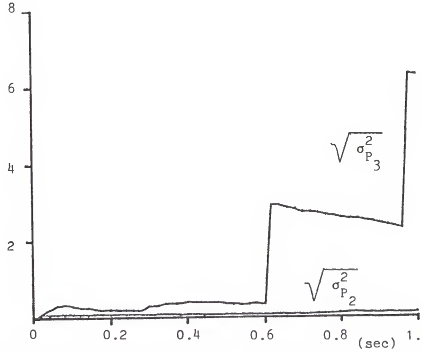

FIGURE 16. Kwakernaak example, Unconstrained, Two sigma jump variance of second moment for non-linear filters

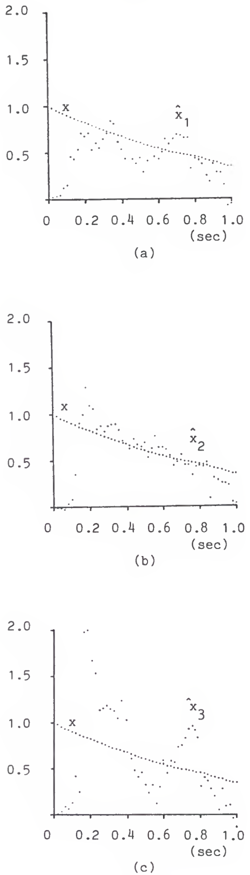

FIGURE 17. Example 3, Unconstrained, One sigma jump linear, 4th and 6th quasimoment filter vs state

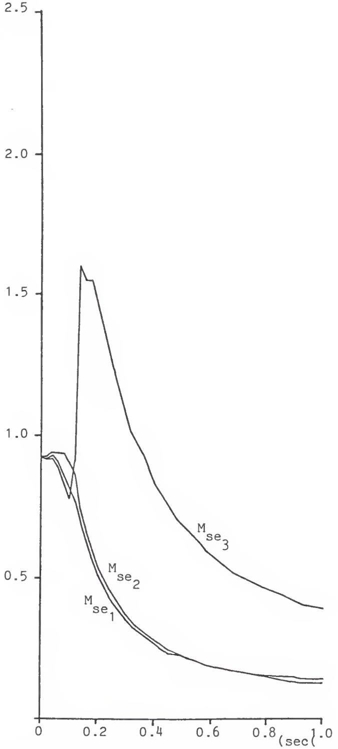

FIGURE 18. Example 3, Unconstrained, One sigma jump mean squared error all filters

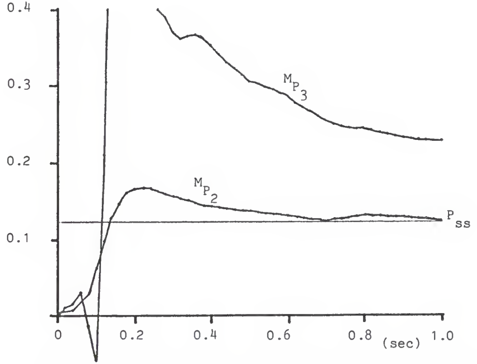

FIGURE 19. Example 3, Unconstrained, One sigma jump mean of second moment for non-linear filters

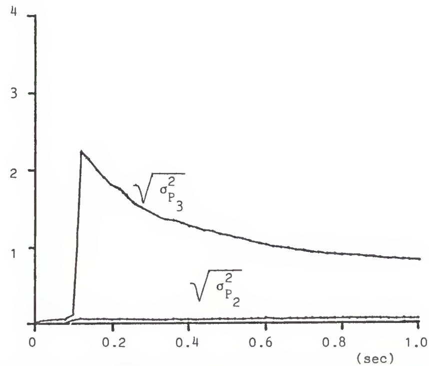

FIGURE 20. Example 3, Unconstrained, One sigma jump variance of second moment for non-linear filters

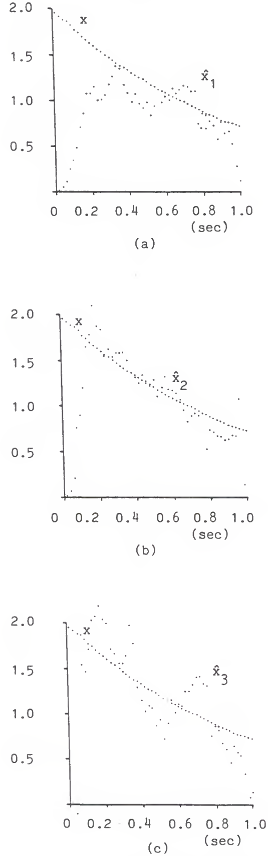

FIGURE 21. Example 3, Unconstrained, Two sigma jump linear, 4th and 6th quasimoment filter vs state

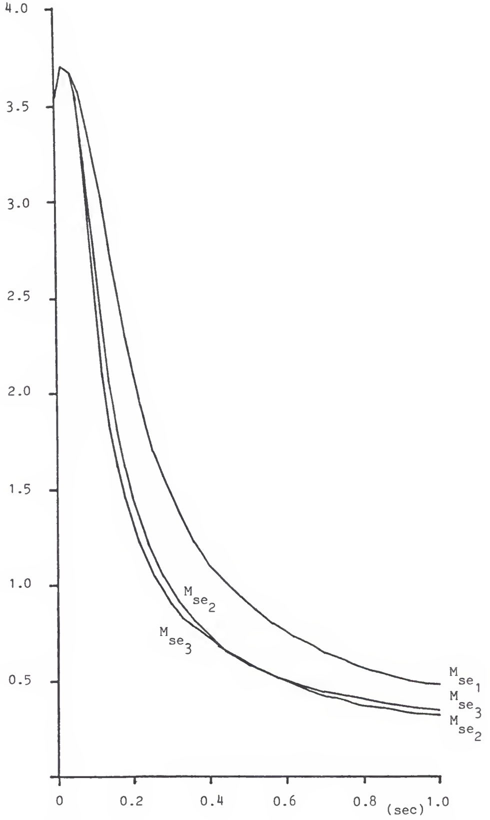

FIGURE 22. Example 3, Unconstrained, Two sigma jump mean squared error for all filters

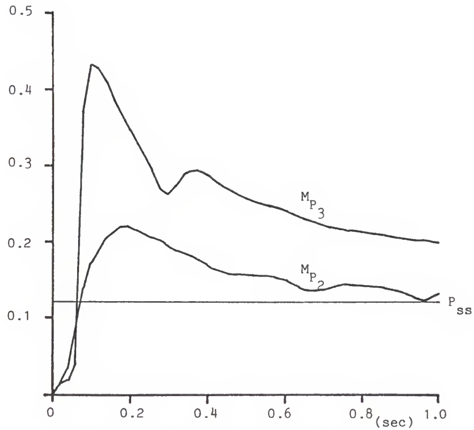

FIGURE 23. Example 3, Unconstrained, Two sigma jump mean of second moment for non-linear filters

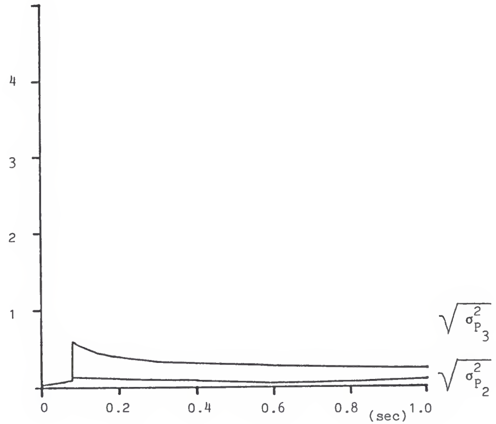

FIGURE 24. Example 3, Unconstrained, Two sigma jump variance of second moment for non-linear filters

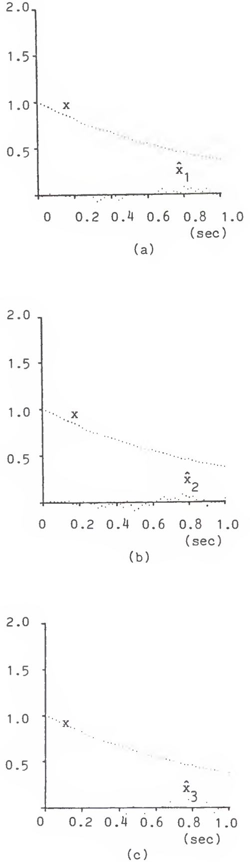

FIGURE 25. Example 4, Unconstrained, One sigma jump linear, 4th and 6th quasimoment filter vs state

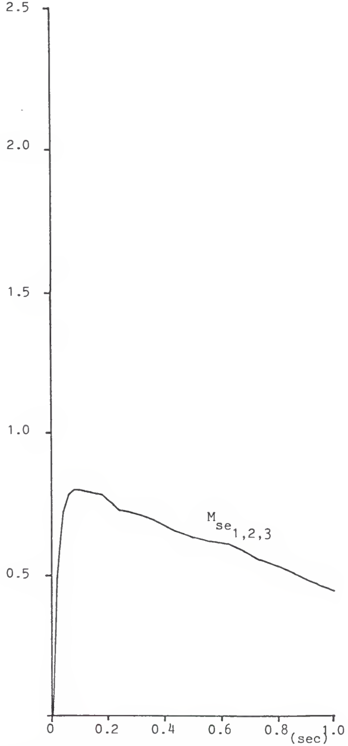

FIGURE 26. Example 4, Unconstrained, One sigma jump mean squared error for all filters

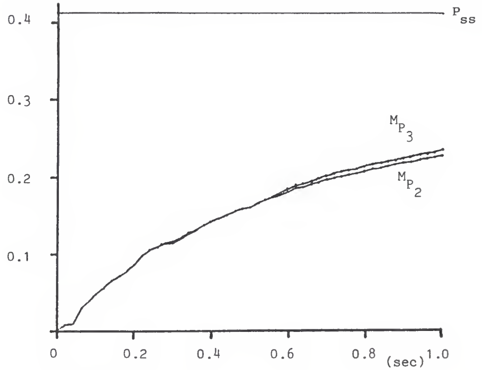

FIGURE 27. Example 4, Unconstrained, One sigma jump mean of second moment for non-linear filters

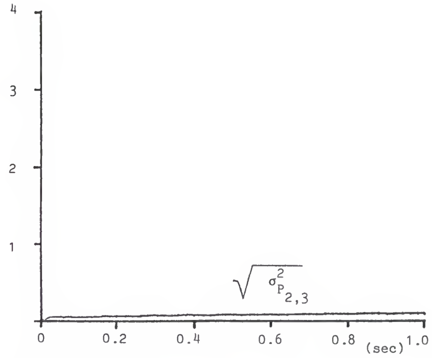

FIGURE 28. Example 4, Unconstrained, One sigma jump variance of second moment for non-linear filters

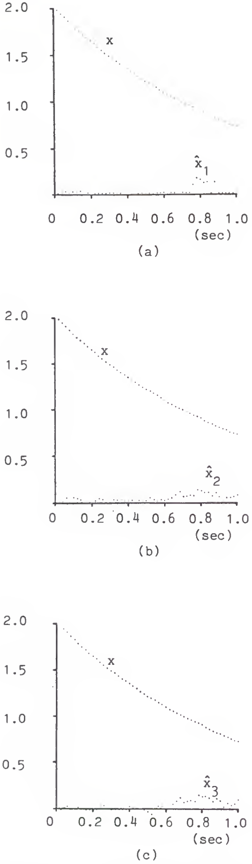

FIGURE 29. Example 4, Unconstrained, Two sigma jump linear, 4th and 6th quasimoment filter vs state

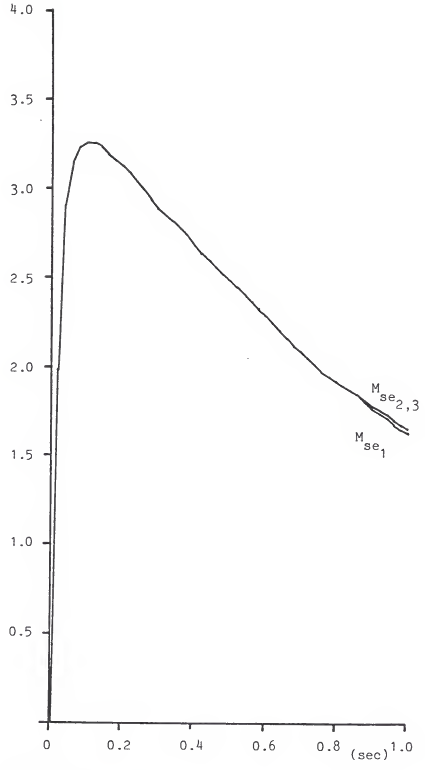

FIGURE 30. Example 4, Unconstrained, Two sigma jump mean squared error for all filters

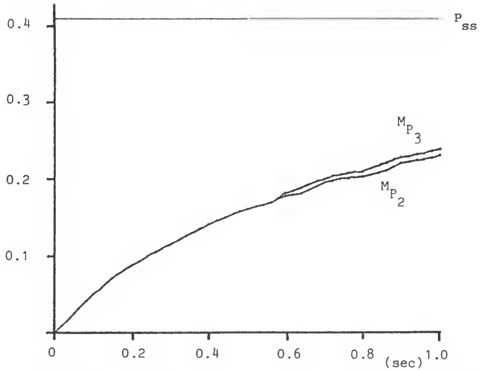

FIGURE 31. Example 4, Unconstrained, Two sigma jump mean of second moment for non-linear filters

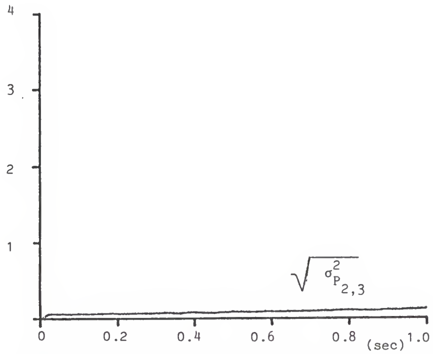

FIGURE 32. Example 4, Unconstrained, Two sigma jump variance of second moment for non-linear filters

performance. This is apparently the result of its increased response over the linear filter and its apparent better stability than the sixth quasimoment filter.

In the fourth quasimoment filter, for all the high signal-to-noise examples, the mean of the second moment increases with increasing jump amplitudes. The variance of the second moment about the mean does not change significantly with jump amplitude and maintains a fairly constant value after the initial response. The mean of the second moment converges to the value of the stationary linear filter error variance as computed by equation (51).

The sixth quasimoment filter demonstrates erratic behavior for jump amplitudes at the one sigma level. This behavior is consistent with a non-increasing mean of the second moment in the initial response and a large variance of the second moment about its mean value relative to the mean. The situation changes with jump amplitudes at the two sigma level. The variance of the second moment is relatively smaller and exhibits a more uniform increase of the mean of the second moment to a lower value than the one sigma cases. This filter demonstrates better performance for larger jump amplitudes and has poorer performance relative to the linear and fourth quasimoment filters for lower jump amplitudes.

In summary for the unconstrained second moment in the filtering equations, the improved performance potential of non-linear filtering equations exists with systems characterized by high signal-to-noise. With high signal-to-noise, the fourth quasimoment filter exhibits improved performance relative to the linear filter. The sixth

quasimoment filtering equation exhibits a higher degree of momentary instability at low values of jump amplitudes than the fourth quasimoment filter with associated degraded performance. At this point, the fourth quasimoment filtering equations yield the best performance.

Constrained Second Moment Comparisons

For the Kwakernaak example and the third example, the following figures illustrate the results of the simulations when the second moment is constrained to be positive semi-definite by the suboptimal process described earlier. It is seen that the mean squared error for the fourth quasimoment filters is very similar to the unconstrained cases presented earlier, however, with a slightly improved result. The mean squared error for the sixth quasimoment filtering equations exhibits a dramatically improved result for the one sigma jump amplitudes with the results from the two sigma jumps also improved.

The variance of the second about its mean is similarly dramatically reduced for the sixth quasimoment filtering equations. The variance of the second moment about its mean for the fourth quasimoment filters is only slightly improved.

The results from implementing the constraint that the second moment remain positive semi-definite yield improved filtering performance recompared to the unconstrained situation.

Summary of Unconstrained and Constrained Examples

Presented in Table 4 are the values of the mean squared error for the linear, fourth quasimoment and sixth quasimoment filters for

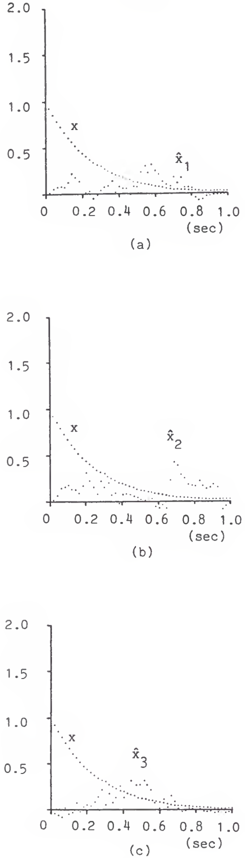

FIGURE 33. Kwakernaak example, Constrained, One sigma jump linear, 4th and 6th quasimoment filter vs. state

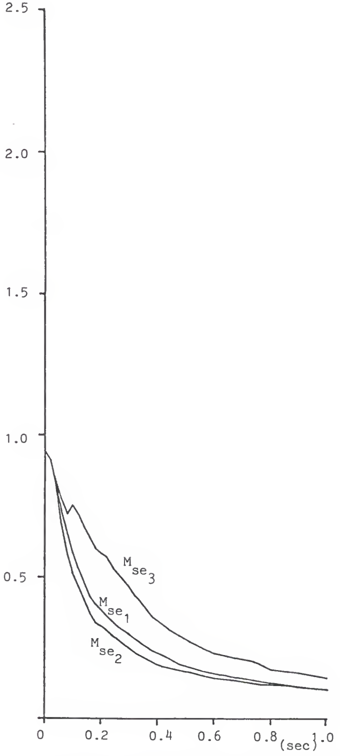

FIGURE 34. Kwakernaak example, Constrained, One sigma jump mean squared error for all filters

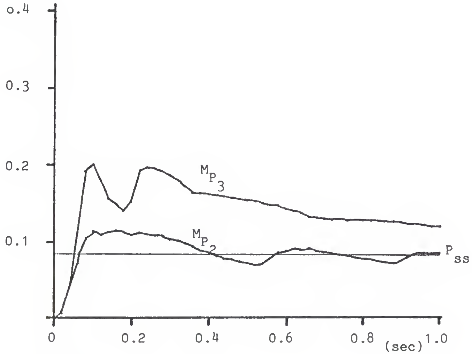

FIGURE 35. Kwakernaak example, Constrained, One sigma jump mean of second moment for non-linear filters

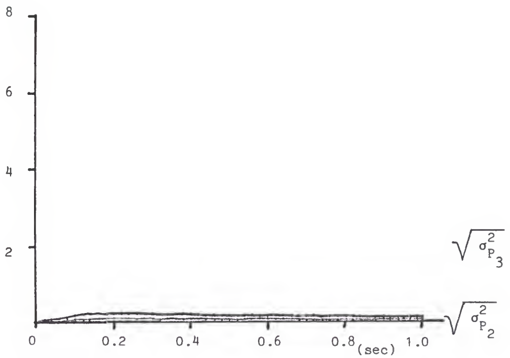

FIGURE 36. Kwakernaak example, Constrained, One sigma jump variance of second moment for non-linear filters

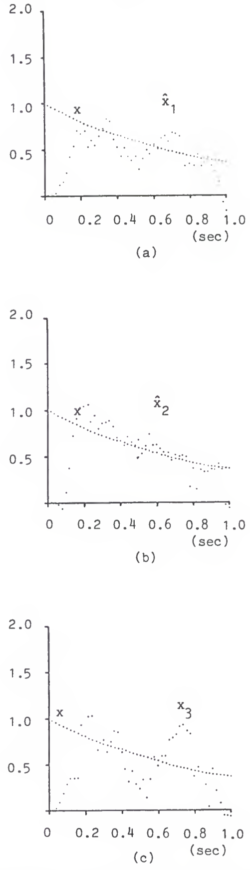

FIGURE 37. Example 3, Constrained, One sigma jump linear, 4th and 6th quasimoment filter vs state

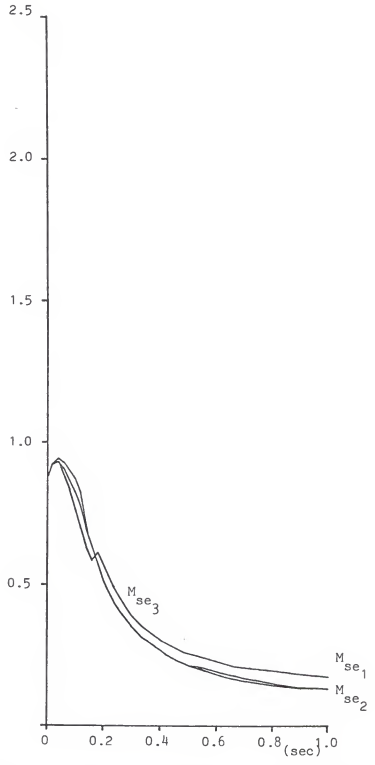

FIGURE 38. Example 3, Constrained, One sigma jump mean square error for all filters

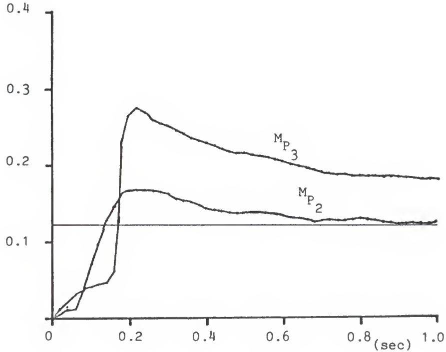

FIGURE 39. Example 3, Constrained, One sigma jump mean of second moment for non-linear filters

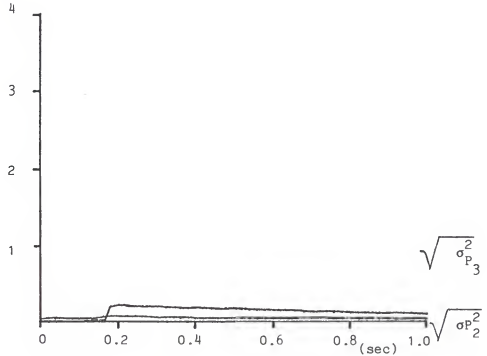

FIGURE 40. Example 3, Constrained, One sigma jump variance of second moment for non-linear filters

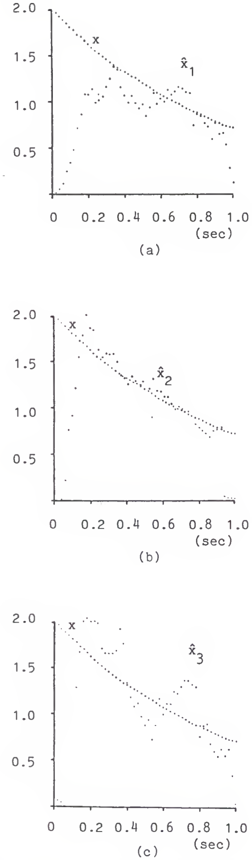

FIGURE 41. Example 3, Constrained, Two sigma jump linear, 4th and 6th quasimoment filter vs state

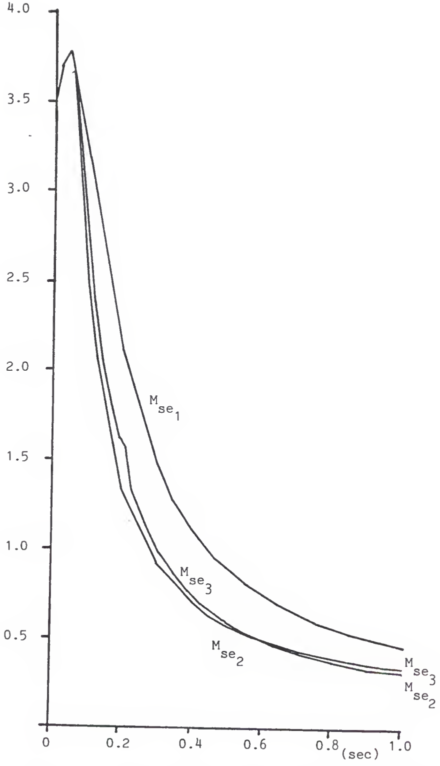

FIGURE 42. Example 3, Constrained, Two sigma jump mean squared error for all filters

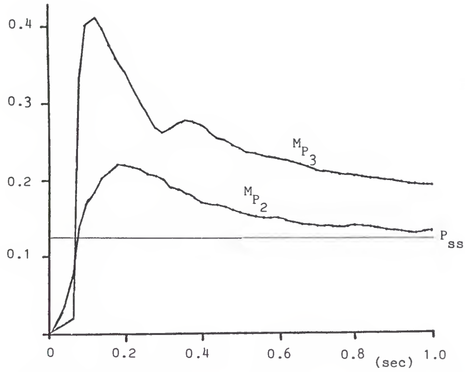

FIGURE 43. Example 3, Constrained, Two sigma jump mean of second moment for non-linear filters

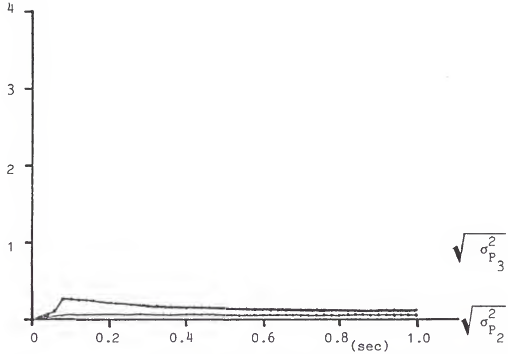

FIGURE 44. Example 3, Constrained, Two sigma jump variance of second moment for non-linear filters

TABLE 4

MEAN SQUARED ERROR AT TIME = $1/\lambda$

Jump Amplitude/Filter		Kwakernaak P unconstrained	Kwakernaak $P \geq 0$	Example 3 P unconstrained	Example 3 $P \geq 0$
1-sigma	linear	.10207		.12950	
	4 quasi	.11615	.09834	.13389	.12384
	6 quasi	.67554	.15240	.39601	.18129
2-sigma	linear	.33902		.46967	
	4 quasi	.24135		.31891	.31030
	6 quasi	.31930		.34411	.33858

the Kwakernaak and third examples for both the unconstrained and constrained second moment situations. These results are at the time corresponding to the inverse of the impulse occurrence rate and quantify the degree of improvement with numbers rather than visualization from the figures.

It is seen that improvements in the simulation results are obtained when the suboptimal constraint is used in all the filtering equations.

The single pulse technique of evaluating the filter's performance presented thus far is not truly reflective of a compound Poisson process driving noise. That is, the jumps were not randomly occurring in time or at random amplitudes. However, the single pulse technique does provide a means of evaluating the relative performance of the filters examined. To confirm the observations from the single pulse results, the Kwakernaak numerical example is used in simulations where the jumps are allowed to occur randomly in time and with random amplitudes. The algorithm described in Snyder (1975) is used in generating the Poisson process time dependency.

Shown in the figures are the results of the linear and fourth quasimoment filters using three values of the "seed" for the random number generator. The second moment for the fourth quasimoment filter is constrained to be positive semi-definite. These results for multiple pulse occurrances again demonstrates that the fourth quasimoment filter exhibits better performance than the linear filter.

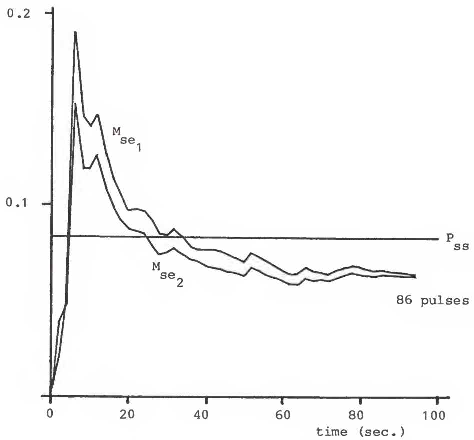

FIGURE 45. Kwakernaak example, Constrained, random jump amplitude and occurrence mean squared error for linear and 4th quasimoment filter $\Delta t = 1.0E-4$: RSEED 1.0E+3

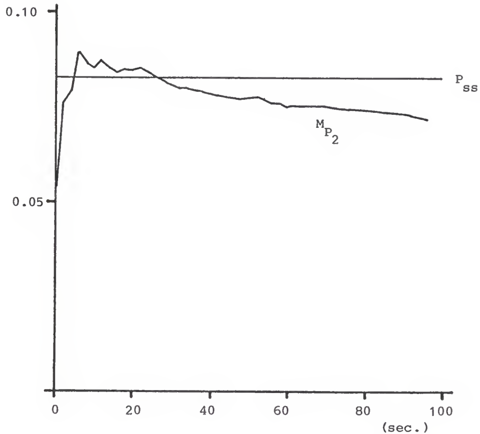

FIGURE 46. Kwakernaak example, Constrained, random jump amplitude and occurrence mean of second moment for linear and 4th quasimoment filter $\Delta t = 1.0E-4$: RSEED 1.0E+3

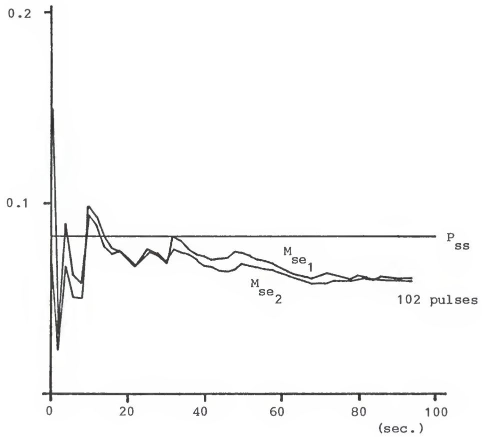

FIGURE 47. Kwakernaak example, Constrained, random jump amplitude and occurrence mean squared error for linear and 4th quasimoment filter $\Delta t = 1.0E-4$: RSEED 1.0E+7

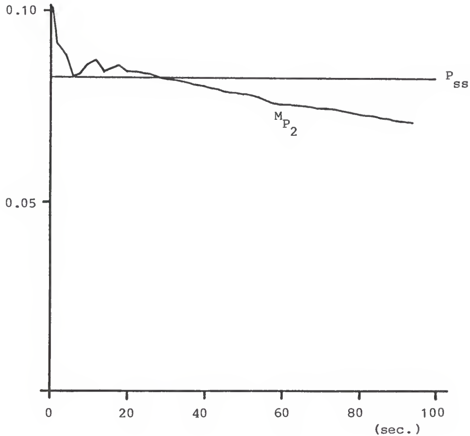

FIGURE 48. Kwakernaak example, Constrained, random jump amplitude and occurrence mean of second moment for linear and 4th quasimoment filter $\Delta t= 1.0E-4$: RSEED 1.0E+7

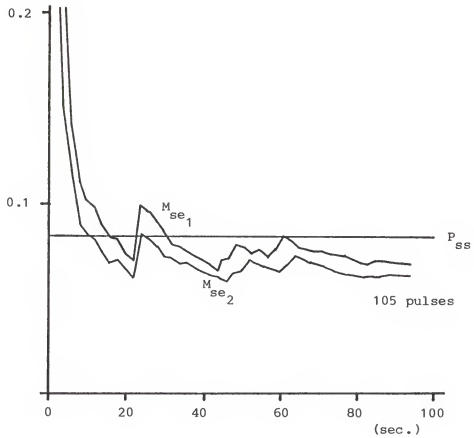

FIGURE 49. Kwakernaak example, Constrained, random jump amplitude and occurrence mean squared error for linear and 4th quasimoment filter Δt=1.0E-4: RSEED 2.0E+7

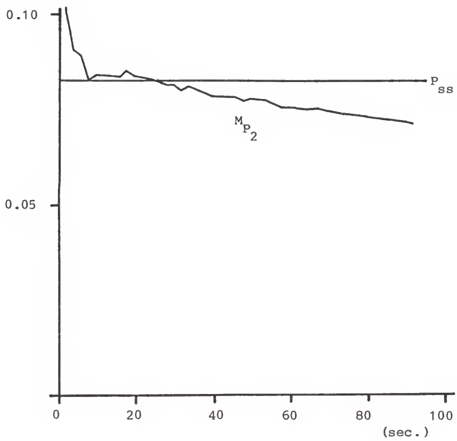

FIGURE 50. Kwakernaak example, Constrained, random jump amplitude and occurrence mean of second moment for linear and 4th quasimoment filter Δt=1.0E-4: RSEED 2.0E+7

Summary

In this chapter, simulation algorithms and data evaluation techniques were established for comparing the performance results for computer implementations of the filtering equations presented in Chapter Two. Based on the stability constraint developed in Chapter Three and implemented in a suboptimal technique, improved performance in the simulated results was demonstrated relative to the unconstrained situation. It was demonstrated, from several numerical examples, that improved performance over the linear filter is obtainable with the non-linear filtering equations when the constraint that the second moment be positive semi-definite was invoked. The fourth quasimoment filtering equations, with the stability constraint exhibit the best performance, which from an implementation consideration is a potential constrain itself.

CHAPTER FIVE
UNSTABLE SYSTEM FILTERING

Introduction

The analysis of the performance of the truncated filtering equations presented in the previous chapters considered only stable systems. It was this type system examined in the previously published works of Eckberg (1970) and Kwakernaak (1975). The intended application of this work is the aerospace application for tracking maneuvering targets referred to in Chapter One. For this problem the system can be considered as neutrally stable for finite times. That is, the system matrix, for the multidimensional system, has some zeros for eigenvalues. In this case, the process noise or "signal" would be bounded only for finite times. To address the neutrally stable problem, the scalar unstable system will be examined as the more severe problem for the performance of the truncated filtering equations.

The performance of the truncated filtering equations when the system given by equation (1) is unstable, $a > 0$, will be examined in the following. First, as in the previous chapters, the stability of these equations will be examined using Krasovskii's Theorem. The results will again establish sufficient conditions for the truncated filtering equations about an equilibrium point. It will be shown that a condition for stability is that the second moment be positive definite.

Given this condition, numerical examples will be presented demonstrating the truncated filters' performance. For this demonstration, the numerical examples of Kwakernaak and Example 3 before will be used with the sign of "a" reversed. This will illustrate two levels of "instability." Only the truncated linear and fourth quasimoment filtering equations will be used for this demonstration. It will be shown that with the developed constraint that the fourth quasimoment filter exhibits improved performance over the linear filter.

Truncated Filtering Equation Stability and Performance-Unstable Systems

For the case of an unstable system, the state estimate, $\hat{x}$, is not an equilibrium point to satisfy the Lyapunov function requirements. If the system is unstable, x and $\hat{x}$ will both become unbounded since there is a non-zero probability that the time interval between jumps will be sufficiently long to effectively make the system unbounded. However, the estimate error should remain bounded and satisfy the requirements for the equilibrium condition. Therefore, the truncated filtering equations will be transformed into a system of equations for the estimate error and higher moments coupled by this error rather than by the state estimate.

Defining the estimate error, e, as

$$e \triangleq \hat{x} - x, \qquad 106$$

the differential error becomes

$$de = aedt + \frac{Pc}{r}(dv - cedt) - d\sigma. \qquad 107$$

Proceeding as before, the unforced equations, written in terms of the estimate error, are

$$\dot{e} = (a - \frac{Pc^2}{r})e \qquad 108$$

$$\dot{P} = 2aP - \frac{P^2c^2}{r} + \lambda\alpha_2 - \frac{k_3c^2}{r} e \qquad 109$$

$$\dot{k}_3 = 3ak_3 - 3\frac{k_3Pc^2}{r} - \frac{k_4c^2}{r} e. \qquad 110$$

The stability of these equations about an equilibrium point are to be evaluated. To satisfy the requirement for the equilibrium point for the Lyapunov function, $\underline{f}(\underline{0}) = \underline{0}$ and $\underline{x} = \underline{0}$ is the equilibrium point, these equations must be transformed as before. For the unstable system, the positive definite steady state value of the second moment is given by

$$P_{ss} = \frac{ar}{c^2}[1 + \sqrt{1 + \frac{\lambda\alpha_2c^2}{a^2r}}]. \qquad 111$$

The resulting transformed unforced equations become

$$\dot{e} = (\tilde{a} - \frac{\tilde{P}c^2}{r} \qquad 112$$

$$\dot{\tilde{P}} = 2\tilde{a}\tilde{P} - \frac{\tilde{P}^2c^2}{r} - \frac{k_3c^2}{r} e \qquad 113$$

$$\dot{k}_3 = 3(a - \frac{Pc^2}{r}) k_3 - \frac{k_4c^2}{r} e \qquad 114$$

where $\tilde{a}$ is given in equation (70). Continuing with the Lyapunov function construction technique given by Krasovskii's Theorem and the evaluation of the negative definiteness of the matrix $F(x)$ using Sylvester's criteria yields the following inequalities as sufficient conditions for stability

$$2(\tilde{a} - \frac{\tilde{P}c^2}{r}) < 0 \qquad 115$$

$$8(\tilde{a} - \frac{\tilde{P}c^2}{r}) - (e + k_3)^2 \frac{c^4}{r^2} > 0 \qquad 116$$

$$2(\tilde{a} - \frac{\tilde{P}c^2}{r})[24(\tilde{a} - \frac{\tilde{P}c^2}{r}) - (e + 3k_3)^2 \frac{c^4}{r^2}]$$

$$+ (e + k_3) \frac{c^2}{r} [-6(e + k_3) \frac{c^2}{r}(\tilde{a} - \frac{\tilde{P}c^2}{r}) - k_4(e + 3k_3) \frac{c^4}{r^2}]$$

$$- k_4 \frac{c^2}{r}[(e + k_3)(e + 3k_3) \frac{c^4}{r^2} + 4k_4(\tilde{a} - \frac{\tilde{P}c^2}{r}) \frac{c^2}{r}] < 0. \qquad 117$$

Or, under equilibrium conditions, these inequalities are equivalent to the following

$$(a - \frac{Pc^2}{r}) < 0 \qquad 118$$

$$(a - \frac{Pc^2}{r})^2 > 0 \qquad 119$$

$$12(a - \frac{Pc^2}{r})^2 < \frac{k_4^2 c^4}{r^2} . \qquad 120$$

For an unstable system, the first of these inequalities return with the initial assumption that the second moment, P, be positive definite. The second of these is always true and the third requires that P not be arbitrarily large. Using additional quasimoment equations in this stability analysis will result in conditions which must be satisfied in addition to those already determined at this point.

Assuming stability, the stationary quasimoments will be examined under limiting conditions as before to establish the degree of coupling between the filtering equations. The stationary quasimoments for an unstable system are

$$\bar{k}_3 = 0 \qquad 121$$

$$\bar{k}_4 = \frac{\frac{3}{4}\lambda\alpha_2^{\,2}}{a\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}} \qquad 122$$

$$\bar{k}_5 = \frac{\frac{9}{8}\lambda\alpha_2^{\,2}\left(1 + \sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}\right)}{-a\left(1 + \frac{\lambda\alpha_2 c^2}{a^2 r}\right)} \qquad 123$$

$$\bar{k}_6 = \frac{\frac{15}{6}\lambda\alpha_2^{\,3}}{a\sqrt{1 + \frac{\lambda\alpha_2 c^2}{a^2 r}}}\left[1 - \frac{9}{32}\lambda\,\frac{\frac{\lambda\alpha_2 c^2}{a^2 r}}{\left(1 + \frac{\lambda\alpha_2 c^2}{a^2 r}\right)}\right]. \qquad 124$$

Comparing these equations with equations (83), (84) and (85), it can be seen that the results of limiting arguments such as those used previously would lead to similar conclusions about the significance of the signal-to-noise ratio since the equations, except for sign differences, are similar. There is a difference, however, in that the process noise signal given before in eqaution (58) does not have a corresponding constant stationary value for the unstable system problem. In this case the process noise variance grows with time.

Numerical Evaluation

The numerical evaluation of the truncated filtering equations will be conducted in this section in a manner similar to that used in Chapter Four. The numerical algorithm presented in equation (102) will again be used to propagate the moment equations in time. The recursive equations (103), (104) and (105) will again be used to provide "running" mean squared error, mean of the second moment, and variance of the second moment respectively.

First the single pulse evaluation for one and two sigma jump levels will be presented for the redefined unstable Kwakernaak example with the constraint of positive definiteness of P imposed. Then for the redefined unstable Kwakernaak and Example 3 examples, a multipulse evaluation will be presented in which the occurrence and amplitude of the jumps are governed by their random occurrence and amplitude behavior.

From the figures for the single pulse responses at both jump levels, it is seen that the fourth quasimoment filter exhibits much more significantly improved performance over the linear filter, in terms of lower mean squared error, for the unstable problem than for the stable problem. It is seen also that the mean of the second moment converges to the stationary value for the unstable system problem as it did for the stable problem. This is consistent with the stationary third quasimoment being zero.

The multipulse performance results also reflect improved performance of the fourth quasimoment filter over that of the linear filter. For both the redefined Kwakernaak and Example 3 examples, the fourth quasimoment filter exhibits lower mean squared error than the linear

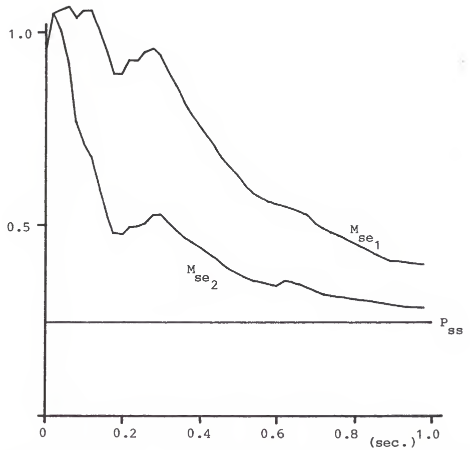

FIGURE 51. Redefineä Kwakernaak, Constrained, One sigma jump mean squared error for linear and 4th quasimoment filter, Δt=1.0E-4

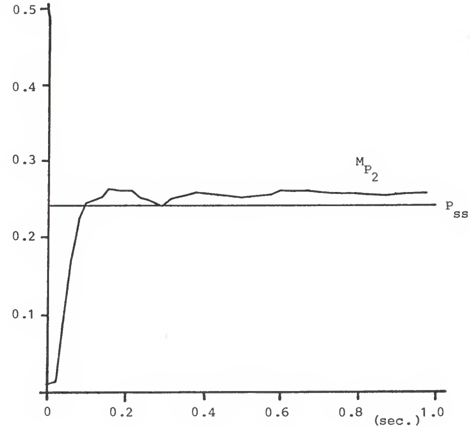

FIGURE 52. Redefined Kwakernaak, Constrained, One sigma jump mean of second moment for linear and 4th quasimoment filter, Δt=1.0E-4

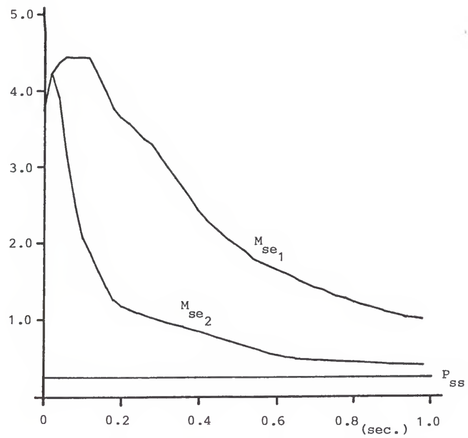

FIGURE 53. Redefined Kwakernaak, Constrained, Two sigma jump mean squared error for linear and 4th quasimoment filter, Δt=1.0E-4

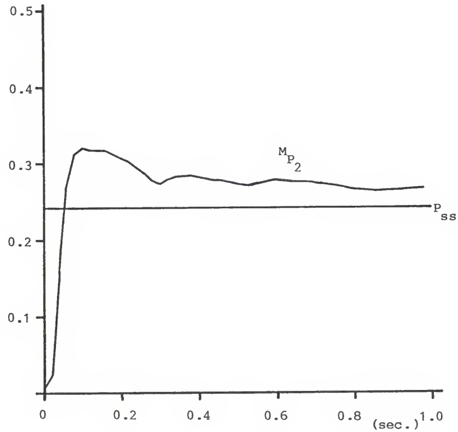

FIGURE 54. Redefined Kwakernaak, Constrained, Two sigma jump mean of second moment for linear and 4th quasimoment filter, $\Delta t=1.0E-4$

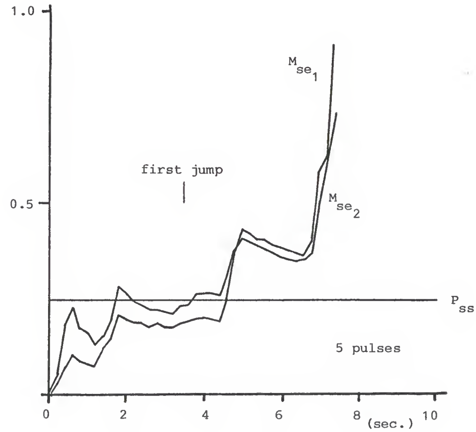

FIGURE 55. Redefined Kwakernaak, Constrained, random, jump amplitude and occurrence mean squared error for linear and 4th quasimoment filter, Δt=1.0E-4

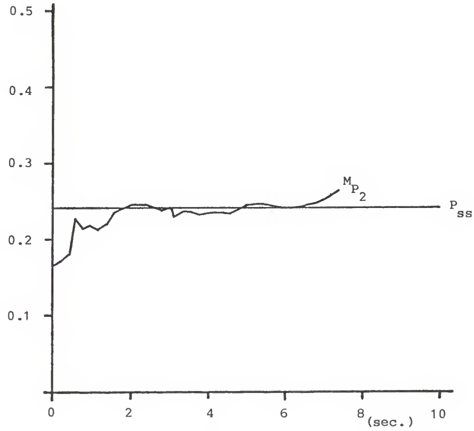

FIGURE 56. Redefined Kwakernaak, Constrained, random, jump amplitude and occurrence mean of second moment for linear and 4th quasimoment filter, Δt=1.0E-4

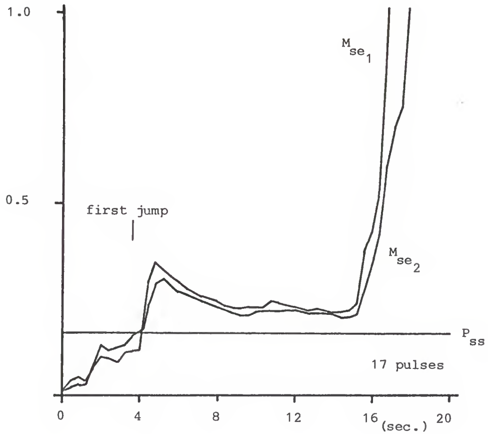

FIGURE 57. Redefined, Example 3, Constrained, random jump amplitude and occurrence mean squared error for linear and 4th quasimoment filter, Δt=1.0E-4

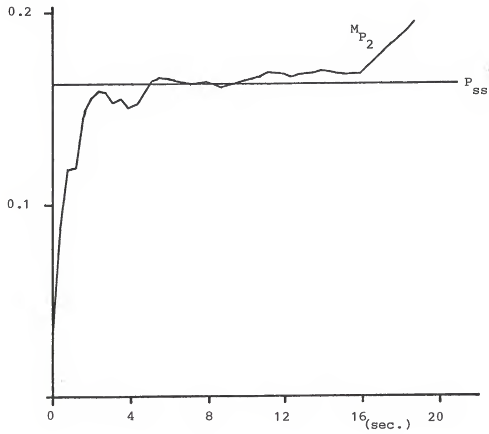

FIGURE 58. Redefined, Example 3, Constrained, random jump amplitude and occurrence mean of second moment for linear and 4th quasimoment filter, Δt=1.0E-4

filter after jumps start occurring. Prior to the first jump, both filters' mean squared error are lower than that expected and compared to the stationary second moment value. Shortly after the first jump occurs, the mean squared error for the fourth quasimoment filter becomes less than that of the linear filter.

As seen from the figures of the multipulse evaluations, a divergence occurs at some point in the sample path. These divergences are the result of numerical imprecisions from taking differences between large numbers. This divergence occurs earlier for the redefined Kwakernaak example than for the redefined Example 3 since the redefined Kwakernaak example is more "unstable" than the redefined Example 3.

Summary

In this chapter the performance of the truncated filtering equations for an unstable system was evaluated. An unstable system possesses a growing process noise or signal variance as does a neutrally stable system in which the potential application for this type of filtering is intended.

It was shown that a condition for the stability of the truncated filtering equations for an unstable system is that the second moment be positive definite. The associated stationary quasimoments were seen to be similar except for signs to those for the stable problem, and that subsequent limiting arguments as to the significance of the signal-to-noise ratio would also produce similar results.

The redefined numerical examples were used to demonstrate the relative performances of the truncated filtering equations for the unstable system problem. It was demonstrated by single pulse and multipulse analyses that the fourth quasimoment filter exhibits improved performance over that of the linear filter. Additionally, the potentially unbounded behavior of an unstable system driven by compound Poisson processes was demonstrated by these numerical examples. For the aerospace application of maneuvering target tracking by a converging missile the duration of the problem will be bounded, thus minimizing the number of jumps, and potential for divergence.

CHAPTER SIX
CONCLUSIONS AND RECOMMENDATIONS

Summary of Conclusions

In this dissertation, optimal filtering equations, using quasi-moment functions, for systems driven by compound Poisson processes have been derived. These equations agree with those obtained by Fisher (1967) for multivariable systems.

The performance of sets of truncated filtering equations was evaluated in terms of responsiveness and stability. It was shown that under low signal-to-noise conditions the linear filter performs poorly and the truncated non-linear filtering equations decouple to become linear with the associated poor performance. To realize the potential improved performance obtainable from the non-linear filtering equations, the stability of these was examined. It was shown that if the system described by equation (1) is stable, then the second moment must be positive semi-definite and, for an unstable system, the second moment must be positive definite.

Numerical evaluations of the performance of these truncated filtering equations was conducted using examples previously published as well as two additional examples. These examples demonstrated the potential for improved performance by using the non-linear filtering equations; however, the stability of these non-linear filtering equations must be assured using the established constraints in order to realize the improvement potential. The numerical examples

also demonstrate the importance of high signal-to-noise to the filter's performance.

Recommendations for Further Research

The problem addressed in this research is a simple time-invariant scalar system, a single state variable and measurement. The potential application will require filter states containing three or four state elements, and also be time-varying. This research should be extended to evaluate the improvements possible for multivariable time-varying systems. With the associated increased dimensionality of the filtering equations, and with the constraints of implementing these equations within a microprocessor, techniques should be examined to propagate the states and moments more efficiently. This may require linearizing the filtering equations and using state transition matrix techniques.

A comparison between the approach used in this research and those of Moose (1975) and Willsky and Jones (1976) should be made to determine the relative performances of the different approaches.

Finally, the sensitivity of these non-linear filtering equations to imperfect modeling and apriori statistical information should be examined. The question of whether these equations exhibit the same sensitivities as does the linear filter needs to be addressed. Also, what techniques can be used to prevent stability and divergence problems are exhibited by linear filtering equations under these conditions.

APPENDIX A

QUASI-MOMENT FUNCTIONS
[Fisher 1967]

In engineering applications where near-Gaussian probability density functions are involved, a good approximation to such density functions which contains only a few parameters is highly desirable.

Let $p(\mu, t)$ be the non-Gaussian density function which has the same mean and variance as $p_g(\mu, t)$. It will be shown in this appendix that the quasi-moment functions are the coefficients in the expansion of the ratio

$$\rho(\mu\ t) = \frac{p(\mu,\ t)}{p_g(\mu,\ t)} \qquad \text{A.1}$$

in a series of Hermite polynomials. If $p(\mu, t)$ is a Gaussian density function, then $\rho(\mu, t) = 1$, and the quasi-moment functions are all zero.

The Hermite polynomials form a complete set of eigenfunctions with a zero mean Gaussian density weighting function over the Euclidean space $-\infty \leq \mu \leq +\infty$. Therefore, the ratio $\rho(\mu, t)$, for any probability density function, whether nearly Gaussian or

$$\int_{-\infty}^{\infty} d\mu \rho^2(\mu,\ t) < \infty \qquad \text{A.2}$$

Also, any $\rho(\mu, t)$ satisfying Equation (A.2) can be approximated to any desired degree of accuracy in the integrable square error sense by

by a finite number of terms in the Hermite polynomial series. In most engineering applications, only a few of the quasi-moment functions are required to give the desired degree of accuracy, and one can assume the remaining higher-order quasi-moment functions to be zero. From this brief discussion, one can see that quasi-moment functions should be very useful in engineering applications, particularly when Gaussian processes and slightly nonlinear systems are involved. The quasi-moment functions will now be developed in some detail.

Let $\varphi(u, t)$ be the characteristic function for $p(\mu, t)$ i.e.,

$$\varphi(u, t) = \int_{-\infty}^{\infty} d\mu p(\mu, t)e^{iu\mu} \qquad \text{A.3}$$

Similarly, let

$$\varphi_g(u, t) = \int_{-\infty}^{\infty} d\mu p_g(\mu, t)e^{iu\mu} \qquad \text{A.4}$$

Denote the mean and variance by $\hat{x}(t)$ and $P(t)$, respectively. Then, by definition of $p_g(\mu, t)$ and $\varphi_g(u, t)$,

$$p_g(\mu, t) = \frac{\exp\{-\frac{1}{2}[\mu - \hat{x}(t)]\ p^{-1}(t)[\mu - \hat{x}(t)]\}}{[(2\pi)|P(t)|]^{\frac{1}{2}}} \qquad \text{A.5}$$

and

$$\varphi_g(u, t) = \exp[iu\ \hat{x}(t) - \tfrac{1}{2}uP(t)u] \qquad \text{A.6}$$

Now, define $k(u, t)$ to be the ratio of $\varphi(u, t)$ and $\varphi_g(u, t)$:

$$\begin{aligned} k(u, t) &= \frac{\varphi(u, t)}{\varphi_g(u, t)} \\ &= \varphi(u, t) \exp[-iu\ \hat{x}(t) + \tfrac{1}{2}u\ P(t)u] \qquad \text{A.7} \end{aligned}$$

Expand $k(u, t)$ in Maclaurin series:

$$k(u, t) = 1 + i\, K_1(t)u_1 + \frac{i^2}{2} K_2(t)u^2 + \frac{i^3}{3} K_3\,(t)u^3 + \ldots$$

$$+ \frac{i^n}{n!} \sum_{n=4}^{\infty} K_n(t)u^n \qquad \text{A.8}$$

where

$$K_n(t) = \left.\frac{1}{i^n} \frac{\partial^n k(u, t)}{\partial u^n}\right|_{u=0} \qquad \text{A.9}$$

The K's are the quasi-moment functions with the subscript denoting the order.

Note that, by the definition of $\hat{x}(t)$ and $P(t)$,

$$\frac{\partial\varphi(u, t)}{u}_{u=0} = i\hat{x}(t) \qquad \text{A.10}$$

and

$$\frac{\partial^2\varphi(u, t)}{\partial u^2}_{u=0} = P(t) + \hat{x}^2(t) \qquad \text{A.11}$$

Therefore, from Equation (A.9) with equations (A.10) and (A.11), it is seen that

$$K_1(t) = 0$$

$$K_2(t) = 0 \qquad \text{A.12}$$

Therefore, all of the quasi-moment functions of first and second order are zero. The mean and the variance effectively replace these functions.

With these definitions the characteristic function $\varphi(\mu, t)$ can be written as follows:

$$\varphi(u, t) = \{\exp[iu\,\hat{x}(t) - \tfrac{1}{2}u\,P(t)u]\} \times \{1 + \sum_{n=3}^{\infty} \frac{i^n}{n!} K(t) u^n\} \qquad A.13$$

Therefore, the characteristic function can be expressed easily and directly in terms of $\hat{x}(t)$, $P(t)$ and the quasi-moment functions.

The probability density $p(\mu, t)$ is obtained by taking the Fourier transform of $\varphi(u, t)$:

$$p(\mu, t) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} du\varphi(u, t) e^{-iu\mu} \qquad A.14$$

Similarly, $p_g(\mu, t)$ is the Fourier transform of $\varphi_g(u, t)$:

$$p_g(\mu, t) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} du \exp\{- iu[\mu - \hat{x}(t)] - \tfrac{1}{2}u\,P(t)u\} \qquad A.15$$

Substitution of Equation (A.13) into Equation (A.14) and utilizing Equation (A.15) yields the interesting result:

$$p(\mu, t) = p_g(\mu, t) + \sum_{n=3}^{\infty} \frac{(-1)^n}{n!} K_n(t) \frac{\partial^n p_g(\mu, t)}{\partial \mu^n} \qquad A.16$$

However, the Hermite polynomials $H_n(\mu, t)$ corresponding to $P^{-1}(t)$ are defined by the following relations:

$$H_n(\mu, t) = (-1)^n \exp[\tfrac{1}{2}\mu\,P(t)\mu] \frac{\partial^n \exp[- \tfrac{1}{2}\mu\,P^{-1}(t)\mu]}{\partial \mu^n}. \qquad A.17$$

Consequently,

$$(-1)^n \frac{\partial^n p_g(\mu, t)}{\partial \mu^n} = H_n[\mu - \hat{x}(t), t] p_g(\mu, t) \qquad A.18$$

Finally, Equation (A.16) may be rewritten with the use of Equation (A.18):

$$p(\mu, t) = p_g(\mu, t) \{1 + \sum_{n=3}^{\infty} \frac{1}{n!} K_n(t) H_n[\mu - \hat{x}(t), t]\} \qquad A.19$$

From Equation (A.1) it is seen that the ratio $\rho(\mu, t)$ is simply the series in the brackets in Equation (A.19). This verifies the earlier statement that the quasi-moment functions are the coefficients in the expansion of $\rho(\mu, t)$ in a series of Hermite polynomials.

The Hermite polynomials can be determined directly from Equation (A.17) rather easily. Therefore, it is clear from Equation (A.19) that the probability density can be expressed easily and directly in terms of $\hat{x}(t)$, $P(t)$ and the quasi-moment functions.

APPENDIX B

DIFFERENTATION RULE AND FILTERING THEOREM
[Kwakernaak 1975]

Let Q_t, $t \geq t_o$, be the semi-martingale defined by

$$dQ_t = F_t dt + dM_t, \qquad t \geq t_o, \tag{B.1}$$

where M is a martingale with respect to a growing family of sigma fields $\mathcal{F}_t$, $t \geq t_o$, and where F is a process that is adapted to $\mathcal{F}$. Let, furthermore, the process Y_t, $t \geq t_o$, be the semi-martingale defined by

$$dY_t = H_t dt + dV_t, \qquad t \geq t_o \tag{B.2}$$

where H is another process that is adapted to $\mathcal{F}$. V is Brownian motion and a martingale with respect to $\mathcal{F}$, such that $E(dV_t\, dV_t) = R(t)dt$ and $R(t) > 0$ for $t \geq t_o$. Define Y_t, $t \geq t_o$, as the growing family of sigma fields generated by the process Y, and let

$$\hat{Q}_t \triangleq E(Q_t | Y_t),$$

$$\hat{F}_t \triangleq E(F_t | Y_t),$$

$$\hat{H}_t \triangleq E(H_t | Y_t),$$

$$\widehat{Q_t H_t} \triangleq E(Q_t H_t | Y_t). \tag{B.3}$$

Fundamental Filtering Theorem for White, Gaussian Observation Noise

The process $\hat{Q}_t$, $t \geq t_o$, satisfies

$$d\hat{Q}_t = \hat{F}_t dt + [\widehat{Q_t H_t} - \hat{Q}_t \hat{H}_t + \hat{S}_t] R^{-1}(t)[dY_t - \hat{H}_t dt], \quad t \geq t_o. \qquad \text{B.4}$$

Here $\hat{S}_t \triangleq E(S_t | Y_t)$, while $S_t \triangleq \frac{d}{dt} \langle M^c, W \rangle_t$. M^c is the continuous part of the martingale M. If M_1 and M_2 are two continuous martingale or semi-martingales, $\langle M_1, M_2 \rangle$ is a stochastic process, which is given by $\langle M_1, M_2 \rangle \triangleq \frac{1}{2}(\langle M_1 + M_2, M_1 + M_2 \rangle - \langle M_1, M_1 \rangle - \langle M_2, M_2 \rangle)$, where $\langle N, N \rangle$ is the increasing process associated with the continuous martingale or semi-martingale N.

Outline of Proof

The proof consists of the following steps. (a) First it is shown that the innovations process I, defined by $dI_t = dY_t - \hat{H}_t dt$, $I_{t_o} = 0$ is a martingale with respect to Y, and, moreover, is a Brownian motion with $d\langle I, I \rangle_t = R(t)dt$. (b) The second step is to prove that the process M^*, defined by $dM^* = d\hat{Q}_t - \hat{F}_t dt$, $M^*_{t_o} = 0$, is also a martingale with respect to Y.

(c) Next, a representation theorem is invoked, which says that if M^* is a martingale with respect to Y, while Y is generated by the semi-martingale

$$\int_{t_o}^{t} \hat{H}_s ds + I_t, \qquad \text{B.5}$$

with I Brownian motion, M^* may be written as

$$M^*_t = M^*_{t_o} + \int_{t_o}^{t} L_s dI_s, \qquad \text{B.6}$$

where L is a process that is adapted to Y. Rewriting (B.6) in differential form it follows that $d\hat{Q}_t = \hat{F}_t dt + L_t dI_t$. (d) It finally is established that L may be expressed as in (B.4) by proving and using the equality $E(d(Q_t|Y_t)\ Y_t) = E(d(\hat{Q}_t\ Y_t|Y_t)$.

In step (d) of the proof, use is made of the differentiation rule of Doleans-Dade and Meyer for discontinuous semi-martingales. Since this rule will be used repeatedly in the paper, it is restated here. Let X_t be discontinuous semi-martingale, and let φ be a twice differentiable function. Then the process $Q_t = \varphi(X_t)$ is also a semi-martingale, such that

$$dQ_t = \varphi_x(X_{t-})dX_t + \tfrac{1}{2}[\varphi_{xx}(X_{t-})d\langle X^c,X^c\rangle_t] \qquad \text{B.7}$$
$$+ d \sum_{t_o \le s \le t} [\varphi(X_s) - \varphi(X_{s-}) - \varphi_x(X_{s-})(X_s - X_{s-})].$$

Here φ_x is the gradient of the function φ, φ_{xx} its Jacobian, while X^c is the continuous part of X, and the summation is carried out over those values of s where X jumps. This differentiation rule reduces to the Ito rule when the process is continuous.

APPENDIX C

COMPUTER PROGRAM LISTINGS

```
C
C     PROGRAM GASPOS
C
      COMMON / CLOCK / TIME, KPLS, VSS, PSS
      COMMON / PROUT / X, Z, V, PLS
      COMMON / PARAM / A, DT
      COMMON / FLTOT1 / DXHAT1, XHAT1, P1
      COMMON / FLTOT2 / DXHAT2(3), XHAT2(3), XM
      COMMON / FLTOT3 / DXHAT3(5), XHAT3(5), XM3
C
      DATA SIGXI,SIGZI,SIGVI,SGEX1I,SGEX2I,SGEX3I,SG21I,SG31I/8*0.0/
      DATA NPTS / 501000 /
      DATA KOUNT / 0 /
C
      CALL RSEED(1000.0)
C
      DO 100 I = 1, NPTS
C
      CALL PROCES
C
      CALL FILTR1
C
      EX1S = (X - XHAT1)**2
C
      CALL FILTR2
C
      EX2S = (X - XM)**2
C
      CALL FILTR3
C
      EX3S = (X - XM3)**2
C
      IF (I.GT.1) GO TO 50
      XMI = X
      ZMI = Z
      VMI = V
```

```
      EX1SMI = EX1S
      EX2SMI = EX2S
      EX3SMI = EX3S
      X21MI = XHAT2(1)
      X31MI = XHAT3(1)
      GO TO 100
50    CONTINUE
C
      CALL RSTATS ( I, X, XMI, SIGXI, XMIP1, SGXIP1 )
      XMI = XMIP1
      SIGXI = SGXIP1
      CALL RSTATS ( I, Z, ZMI, SIGZI, ZMIP1, SGZIP1 )
      ZMI = ZMIP1
      SIGZI = SGZIP1
      CALL RSTATS ( I, V, VMI, SIGVI, VMIP1, SGVIP1 )
      VMI = VMIP1
      SIGVI = SGVIP1
      CALL RSTATS ( I, EX1S, EX1SMI, SIGE1I, EX1SMP, SGEX1P )
      EX1SMI = EX1SMP
      SIGE1I = SGEX1P
      CALL RSTATS ( I, EX2S, EX2SMI, SIGE2I, EX2SMP, SGEX2P )
      EX2SMI = EX2SMP
      SIGE2I = SGEX2P
      CALL RSTATS ( I, EX3S, EX3SMI, SIGE3I, EX3SMP, SGEX3P )
      EX3SMI = EX3SMP
      SIGE3I = SGEX3P
      CALL RSTATS ( I, XHAT3(1), X31MI, SIG31I, X31MP1, SG31P1 )
      X31MI = X31MP1
      SIG31I = SG31P1
      CALL RSTATS ( I, XHAT2(1), X21MI, SIG21I, X21MP1, SG21P1 )
      X21MI = X21MP1
      SIG21I = SG21P1
C
      KOUNT = KOUNT + 1
      IF(KOUNT - 2000) 100, 90, 90
90    CONTINUE
```

```
C
      KOUNT = 0
C
      WRITE(6,901) TIME, X, Z, V, XHAT1, XM, XM3, XHAT2(1)
     1 ,XHAT3(1), P1
      WRITE(6,902) XMIP1,ZMIP1,VMIP1,EX1SMP,EX2SMP,EX3SMP
     1 ,X21MP1,X31MP1,VSS
      WRITE(6,903) KPLS,SGXIP1,SGZIP1,SGVIP1,SGEX1P,SGEX2P,SGEX3P
     1 ,SG21P1,SG31P1,PSS
C
100   CONTINUE
C
901   FORMAT ( 2X, E12.5, 9E12.5 )
902   FORMAT ( 14X, 9E12.5 )
903   FORMAT ( 4X, I5, 5X, 9E12.5 )
C
      STOP
      END
      SUBROUTINE PROCES
C
      COMMON / STATS / PMEAN,PSIG,VMEAN,VSIG,RINV,Q,RATE
      COMMON / CLOCK / TIME, KPLS, VSS, PSS
      COMMON / PRCESS / DY
      CCMMON / PARAM / A,DT
      COMMCN / PROUT / X, Z, V, PLS
C
      IF (INIT) 5, 5, 7
5     CONTINUE
      INIT = 1
C
      PMEAN = 0.0
      PSIG = SQRT(1.0)
      VMEAN = 0.0
      VSIG = SQRT(0.02)
      RATE = 1.0
C
```

```
      AMP = 2.0*PSIG
C
      A = -4.0
      DT = 0.2E-5
      SQDT = SQRT(DT)
C
      RINV = 1.0/(VSIG**2)
      Q = PSIG**2
C
      VSS = -RATE*Q/(2.0*A)
      R = 1.0/RINV
      PSS = -A*R*(SQRT(1.0+RATE*Q/(R*A**2)) - 1.0)
C
7     CONTINUE
C
      CALL NOISE ( VMEAN,VSIG,V,12 )
C
      CALL POISS ( TIME, IMPLS )
      IF ( IMPLS.LT.1 ) GO TO 10
      TPLS = TIME
      KPLS = KPLS + IMPLS
C
      CALL NOISE ( PMEAN,PSIG,PLS,12 )
      PLS = AMP
      GO TO 20
C
10    CONTINUE
C
      PLS = 0.0
C
20    CONTINUE
C
      DX = A*X*DT + PLS
      DY = X*DT + V*SQDT
C
      TIME = TIME + DT
      X = X + DX
```

```
      Z = DY/DT                                                         PRI01450
C                                                                       PRI01460
      RETURN                                                            PRI01470
      END                                                               PRI01480
      SUBROUTINE POISS ( T, IMPLS )                                     PRI01490
C                                                                       PRI01500
      COMMON / STATS / PMEAN,PSIG,VMEAN,VSIG,RINV,Q,RATE                PRI01510
      COMMON / PARAM / A, DT                                            PRI01520
C                                                                       PRI01530
      EPS = DT/2.0                                                      PRI01540
      TK = 100.0*DT                                                     PRI01550
C     IF (INIT) 3, 3, 5                                                 PRI01560
3     CONTINUE                                                          PRI01570
C                                                                       PRI01580
C     TK = 0.0                                                          PRI01590
C     INIT = 1                                                          PRI01600
C                                                                       PRI01610
5     CONTINUE                                                          PRI01620
C                                                                       PRI01630
      TEST = T - TK                                                     PRI01640
      IF (ABS(TEST).LT.EPS) GO TO 10                                    PRI01650
C     IF (TEST.GT.ZDT) GO TO 10                                         PRI01660
C                                                                       PRI01670
      IMPLS = 0                                                         PRI01680
      GO TO 20                                                          PRI01690
C                                                                       PRI01700
10    CONTINUE                                                          PRI01710
C                                                                       PRI01720
      IMPLS = 1                                                         PRI01730
C     Z = RNDMF(1.0)                                                    PRI01740
C     ZDT = -ALOG(Z)/RATE                                               PRI01750
C     TK = T                                                            PRI01760
C                                                                       PRI01770
20    CONTINUE                                                          PRI01780
C                                                                       PRI01790
      RETURN                                                            PRI01800
```

```
      END
      SUBROUTINE NOISE ( XMEAN, SIGMA, RNDMN, N )
C
      A = 0.0
C
      DO 1 I = 1, N
      Y = RNDMF(1.0)
1     A = A + Y
C
      RNDMN = (A - N*0.5)*SIGMA + XMEAN
C
      RETURN
      END
      SUBROUTINE FILTR1
C
      COMMON / STATS / PMEAN,PSIG,VMEAN,VSIG,RINV,Q,RATE
      COMMON / PARAM / A,DT
      COMMON / PRCESS / DY
      COMMON / FLTOT1 / DXHAT,XHAT,P
C
      DX = A*XHAT*DT + P*RINV*(DY - XHAT*DT)
      DP = (2.0*A*P + RATE*Q - P**2*RINV)*DT
C
      XHAT = XHAT + DX
      P = P + DP
C
      RETURN
      END
      SUBROUTINE FILTR2
C
      DIMENSION F(3,3), D(3,3), TEMP(3), QVEC(3)
C
      COMMON / STATS / PMEAN,PSIG,VMEAN,VSIG,RINV,Q,RATE
      COMMON / PARAM / A,DT
      COMMON / PRCESS / DY
      COMMON / FLTOT2 / DX(3), X(3), XM
```

```
C
      DATA F / 9*0.0 /
      DATA D / 3*0.0,1.0,3*0.0,1.0,0.0 /
      DATA QVEC / 3*0.0 /
C
      P = X(1)
      XK3 = X(2)
      XK4 = X(3)
C
      A1 = -A + P*RINV
      XK3RI = XK3*RINV
C
      RESID = DY - XM*DT
C
      DM = A*XM*DT + P*RINV*RESID
C
      F(1,1) = 2.0*A - P*RINV
      F(2,2) = -3.0*A1
      F(3,3) = -4.0*A1
      F(3,2) = -3.0*XK3RI
C
      QVEC(1) = RATE*Q
      QVEC(3) = 3.0*RATE*Q**2
C
      CALL MULT ( F, X, DX, 3, 3, 1 )
      CALL ADD ( QVEC, DX, DX, 3, 1 )
      CALL SMLT (DT, DX, DX, 3, 1 )
C
      CALL MULT ( D, X, TEMP, 3, 3, 1 )
      CALL SMLT ( RINV, TEMP, TEMP, 3, 1 )
      CALL SMLT ( RESID, TEMP, TEMP, 3, 1 )
      CALL ADD ( DX, TEMP, DX, 3, 1 )
C
      DP = DX(1)
      PTEMP = P + DP
      IF ( A - PTEMP*RINV ) 20, 10, 10
```

```
10    CONTINUE
      DX(1) = 0.0
20    CONTINUE
C
      CALL ADD ( X, DX, X, 3, 1 )
C
      XM = XM + DM
C
      RETURN
      END
      SUBROUTINE FILTR3
C
      DIMENSION F(5,5), D(5,5), TEMP(5), QVEC(5)
C
      COMMON / STATS / PMEAN,PSIG,VMEAN,VSIG,RINV,Q,RATE
      COMMON / PARAM / A,DT
      COMMON / PRCESS / DY
      COMMON / FLTCT3 / DX(5), X(5), XM
C
      DATA F / 25*0.0 /
      DATA D / 5*0.0,1.0,5*0.0,1.0,5*0.0,1.0,5*0.0,1.0,0.0 /
      DATA QVEC / 5*0.0 /
C
      P = X(1)
      XK3 = X(2)
      XK4 = X(3)
      XK5 = X(4)
      XK6 = X(5)
C
      A1 = -A + P*RINV
      XK3RI = XK3*RINV
      XK4RI = XK4*RINV
      XK5RI = XK5*RINV
C
      RESID = DY - XM*DT
C
```

```
      DM = A*XM*DT + P*RINV*RESID                                       PRI02890
C                                                                       PRI02900
      F(1,1) = 2.0*A - P*RINV                                           PRI02910
      F(2,2) = -3.0*A1                                                  PRI02920
      F(3,3) = -4.0*A1                                                  PRI02930
      F(3,2) = -3.0*XK3RI                                               PRI02940
      F(4,4) = -5.0*A1                                                  PRI02950
      F(4,2) = -7.5*XK4RI                                               PRI02960
      F(5,5) = -6.0*A1                                                  PRI02970
      F(5,3) = -7.5*XK4RI                                               PRI02980
      F(5,2) = -3.0*XK5RI                                               PRI02990
C                                                                       PRI03000
      D(4,2) = -10.0*XK3                                                PRI03010
      D(5,2) = -15.0*XK4                                                PRI03020
C                                                                       PRI03030
      QVEC(1) = RATE*Q                                                  PRI03040
      QVEC(3) = 3.0*RATE*Q**2                                           PRI03050
      QVEC(5) = 15.0*RATE*Q**3                                          PRI03060
C                                                                       PRI03070
      CALL MULT ( F, X, DX, 5, 5, 1 )                                   PRI03080
      CALL ADD ( QVEC, DX, DX, 5, 1 )                                   PRI03090
      CALL SMLT (DT, DX, DX, 5, 1 )                                     PRI03100
C                                                                       PRI03110
      CALL MULT ( D, X, TEMP, 5, 5, 1 )                                 PRI03120
      CALL SMLT ( RINV, TEMP, TEMP, 5, 1 )                              PRI03130
      CALL SMLT ( RESID, TEMP, TEMP, 5, 1 )                             PRI03140
      CALL ADD ( DX, TEMP, DX, 5, 1 )                                   PRI03150
C                                                                       PRI03160
      DP = DX(1)                                                        PRI03170
      PTEMP = P + DP                                                    PRI03180
      IF ( A - PTEMP*RINV ) 20, 10, 10                                  PRI03190
10    CONTINUE                                                          PRI03200
      DX(1) = 0.0                                                       PRI03210
20    CONTINUE                                                          PRI03220
C                                                                       PRI03230
      CALL ADD ( X, DX, X, 5, 1 )                                       PRI03240
```

```
C
      XM = XM + DM
C
      RETURN
      END
      SUBROUTINE ADD ( A, B, C, NRA, NCA )
      DIMENSION A(NRA,NCA), B(NRA,NCA), C(NRA,NCA)
C
      DO 10 J = 1, NCA
      DO 10 I = 1, NRA
         C(I,J) = A(I,J) + B(I,J)
10    CONTINUE
C
      RETURN
      END
      SUBROUTINE SUBT ( A, B, C, NRA, NCA )
      DIMENSION A(NRA,NCA), B(NRA,NCA), C(NRA,NCA)
C
      DO 10 J = 1, NCA
      DO 10 I = 1, NRA
         C(I,J) = A(I,J) - B(I,J)
10    CONTINUE
C
      RETURN
      END
      SUBROUTINE MULT ( A, B, C, NRA, NCA, NCB )
C
      DIMENSION A(1), B(1), C(1)
C
      DO 50 I = 1, NRA
         IJ = I
         DO 10 J = 1, NCB
            C(IJ) = 0.0
            IJ = IJ + NRA
10    CONTINUE
      IJ = I
```

```
      DO 40 J = 1, NCA
         IF ( A(IJ).EQ.0.0 ) GO TO 30
         JKN = I
         JKL = J
         DO 20 K = 1, NCB
            C(JKN) = C(JKN) + A(IJ)*B(JKL)
            JKN = JKN + NRA
            JKL = JKL + NCA
20       CONTINUE
30       IJ = IJ + NRA
40    CONTINUE
50    CONTINUE
C
      RETURN
      END
      SUBROUTINE SMLT ( S, A, B, NRA, NCA )
      DIMENSION A(NRA,NCA), B(NRA,NCA)
C
      DO 10 I = 1, NRA
      DO 10 J = 1, NCA
         B(I,J) = S*A(I,J)
10    CONTINUE
C
      RETURN
      END
      SUBROUTINE RSTATS ( I, X, XMI, SIGXI, XMIP1, SGXIP1 )
C
      XI = I
      XIP1 = XI + 1.0
C
      XMIP1 = XMI + ( X - XMI )/XIP1
C
      SGXIP1 = SIGXI + ((XI/XIP1)*( XMI - X )**2 - SIGXI )/XI
C
      RETURN
      END
```

REFERENCES

K. J. Astrom (1965), "On a First-order Stochastic Differential Equation," Int. J. Control, Vol. 1, pp. 301-326.

S. P. Au (1979), "State Estimation for Linear Systems Driven Simultaneously by Wiener and Poisson Processes," Ph.D. Thesis, University of Illinois at Urbana-Champaign.

S. Au and A. H. Haddad (1978), "Suboptimal Sequential Estimation - Detection Scheme for Poisson Driven Linear Sysems," Information Sciences, Vol. 16, pp. 95-113.

S. P. Au, A. H. Haddad, and H. V. Poor (1982), "A State Estimation Algorithm for Linear Systems Driven Simultaneously by Wiener and Poisson Processes," IEEE Trans. Auto. Control, Vol. AC-27, No. 3, pp. 617-626.

R. S. Bucy and P. D. Joseph (1968), Filtering for Stochastic Processes with Applications to Guidance, Interscience, New York.

D. J. Clements and B. D. O. Anderson (1973), "Well-Behaved Ito Equations with Simulations that Always Misbehave," IEEE Trans. Auto. Control, Vol. AC-18, No. 6, pp. 676-677.

A. E. Eckberg, Jr. (1970), "Optimal Estimation of Very-Low Frequency Atmospheric Noise," Engineer's Degree Thesis, MIT.

M. Fiorina and C. Maffezzoni (1979), "A Direct Approach to Jump Detection in Linear Time-Invariant Systems with Application to Power System Perturbation Detection," IEEE Trans. Auto. Control, Vol. AC-24, No. 3, pp. 428-434.

J. R. Fisher (1967), "Optimal Non-linear Filtering," Advances in Control Systems, Theory and Applications, Vol. 5, pp. 197-300.

J. R. Fisher and E. B. Stear (1967), "Optimal Non-linear Filtering for Independent Increment Processes," IEEE Trans. Information Theory, Vol. IT-3, pp. 558-578.

N. H. Gholson and R. L. Moose (1977), "Maneuvering Target Tracking Using Adaptive State Estimation," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-13, No. 3, pp. 310-317.

A. H. Jazwinski (1970), Stochastic Processes and Filtering Theory, Academic Press, New York.

C. D. Johnson (1977), "Theory of Disturbance-Accommodating Controllers," Control and Dynamic Systems - Advances in Theory and Applications, Vol. 13, pp. 387-489.

H. Kwakernaak (1975), "Filtering for Systems Excited by Poisson White Noise, Control Theory, Numerical Methods and Computer Systems Modelling," Lecture Notes in Economics and Mathematical Systems, Vol. 107, Springer-Verlag, pp. 463-492.

H. Kwakernaak (1980), "Estimation of Pulse Heights and Arrival Times," Automatica, Vol. 16, pp. 367-377.

J. H. Laning, Jr. and R. H. Battin (1956), Random Processes in Automatic Control, McGraw-Hill, New York.

J. Mendel (1977), "White-Noise Estimators for Seismic Data Processing in Oil Exploration," IEEE Trans. Auto. Control, Vol. AC-22, No. 5, pp. 694-706.

T. P. McGarty (1974), Stochastic Systems and State Estimation, Wiley and Sons, New York.

R. L. Moose (1975), "An Adaptive State Estimation Solution to the Maneuvering Target Problem," IEEE Trans. Auto. Control, Vol. AC-20, No. 3, pp. 359-362.

R. L. Moose, H. F. Vanlandingham, and D. H. McCabe (1979), "Modeling and Estimation for Tracking Maneuvering Targets," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-15, No. 3, pp. 448-456.

K. Ogata (1967), State Space Analysis of Control Systems, Prentice-Hall, Englewood Cliffs, NJ.

C. F. Price and R. S. Warren (1973), "Performance Evaluation of Homing Guidance Laws for Tactical Missiles," The Analytic Sciences Corporation TR-170-4.

R. M. Rogers (1982), "Sensitivity of Higher Order Homing Guidance Laws to Parameter Variations," Presented at IEEE Southeastcon.

D. L. Snyder (1968), "Optimal Binary Detection of Known Signals in a Non-Gaussian Noise Resembling VLF Atmospheric Noise," Presented at IEEE Wescon.

D. L. Snyder (1975), Random Point Processes, Wiley-Interscience, New York.

A. H. Willsky and H. L. Jones (1976), "A Generalized Likelihood Ratio Approach to the Detection and Estimation of Jumps in Linear Systems," IEEE Trans. Auto. Control. Vol. AC-21, No. 1, pp. 108-112.

W. M. Wonham (1965), "Some Applications of Stochastic Differential Equations to Optimal Non-linear Filtering," SIAM Journal of Control, Vol. 2, No. 3, pp. 347-369.

P. Zarchan (1979), "Complete Statistical Analysis of Non-linear Missile Guidance Systems - SLAM," AIAA Jr. Guidance and Control, Vol. 2, No. 1, pp. 71-78.

BIOGRAPHICAL SKETCH

Robert M. Rogers was born December 17, 1946, in Dothan, Alabama. In June 1964 he was graduated from Malone High School, Malone, Florida. In June 1966, he received an Associate of Arts degree from Chipola Junior College. In September of that year he transferred to the University of Florida and earned a degree of Bachelor of Engineering in Aerospace Engineering with Honors in March 1969. After one and one-half years with Martin Marietta Corporation, he returned to the University of Florida and received a Master of Science degree in the Aerospace Engineering Department in August 1971. Since graduation in 1971 until January 1981, he was employed at Eglin Air Force Base, Florida, and studying part-time at the Northwest Florida Graduate Center on the base. From September 1976 until August 1977 he was enrolled as a student in the Air Force's long-term, full-time program on-campus in Gainesville, Florida. In January 1981, he was employed by The Analytic Sciences Corporation (TASC) in their Fort Walton Beach office, and from October of that year has been employed by System Dynamics Incorporated located in Gainesville.

Robert M. Rogers is married to the former Jackie Henderson and the father of a son, Robbie, and a daughter, Rosalie. He is a member of Phi Kappa Phi, Tau Beta Pi, and Sigma Tau. He is also a member of the American Institute of Aeronautics and Astronautics. He is the recipient of the Air Force Civilian Meritorious Service Award.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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